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Original Article

Weighted Ricci curvature in Riemann-Finsler geometry

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ABSTRACT: Ricci curvature is one of the important geometric quantities in Riemann-Finsler geometry. Together with the S-curvature, one can define a weighted Ricci curvature for a pair of Finsler metric and a volume form on a manifold. One can build up a bridge from Riemannian geometry to Finsler geometry via geodesic fields. Then one can estimate the Laplacian of a distance function and the mean curvature of a metric sphere under a lower weighted Ricci curvature by applying the results in the Riemannian setting. These estimates also give rise to a volume comparison of Bishop-Gromov type for Finsler metric measure manifolds.

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1. Introduction

On a complete Riemannian manifold (M, g) with a volume form $dV = e^{-f}dV_g$, we have the so-called weighted Laplacian Δ_f and weighted Ricci curvature Ric_f^N defined by

$$\Delta_f u = \mathrm{div}(\nabla u) = \Delta u - df(\nabla u),$$

$$\mathrm{Ric}_f^N = \mathrm{Ric}_g + \mathrm{Hess}_g(f) - \frac{1}{N-n}(df)^2.$$

Set $\mathrm{Ric}_f^\infty := \mathrm{Ric}_g + \mathrm{Hess}_g(f)$. In literatures, Ric_f^N is called the *N-Bakery-Emery Ricci tensor* and Ric_f^∞ the *Bakry-Emery Ricci tensor*. In 1997, Z. Qian gave an upper bound on the weighted Laplacian $\Delta_f\rho$ of a distance function $\rho(x) = d(p, x)$ under a lower weighted Ricci curvature bound: $\mathrm{Ric}_f^N \geq (N-1)H$. Using the upper bound $\Delta_f\rho$, he generalized the Bishop-Gromov volume comparison to weighted volume ([8]). Later, G. Wei and W. Wylie gave an estimate on $\Delta_f\rho$ under other Ricci curvature bounds $\mathrm{Ric}_f^\infty \geq (n-1)H$ and $df \geq -\delta$. These results can be applied to Finsler metric measure manifolds after we build up a bridge from the Riemannian setting to the non-Riemannian setting.

Finsler metrics are just Riemannian metrics without quadratic restriction. The notions of Riemann curvature and Ricci curvature in Riemann geometry are naturally extended to Finsler geometry. Every Finsler metric F on a manifold M induces a spray G which is a special vector field on the tangent bundle TM. The geodesics of F are characterized as the projections of the integral curves of G. The Riemann curvature and some other

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non-Riemannian quantities such as the Berwald curvature and the χ-curvature are defined by the spray G. There are quantities such as the Cartan torsion and the Landsberg curvature are defined by the Finsler metric F and its spray G.

On a Finsler metric measure manifold (M, F, dV), the non-linear weighted Laplacian $\Delta = \Delta_{(F,dV)}$ is defined in a natural way:

$$\Delta u = \operatorname{div}(\nabla u),$$

where ∇u denotes the gradient of u with respect to F and $\operatorname{div}(\cdot)$ denotes the divergence of a vector field with respect to the volume form dV.

The weighted Ricci curvature $\mathrm{Ric}^N = \mathrm{Ric}^N_{(F,dV)}$ of (F, dV) are defined by the Ricci curvature Ric of F and the S-curvature S of (F, dV).

$$\mathrm{Ric}^N = \mathrm{Ric} + \dot{S} - \frac{1}{N-n} S^2.$$

This weighted Ricci curvature is first studied by S. Ohta [6].

My motivation to write this survey article is to build up a bridge from Riemannian geometry to non-Riemannian geometry via geodesic fields. Then many comparison theorems on Riemannian manifolds with a volume form can be carried over to Finsler manifolds wit a volume form. This goal can be achieved due to the fact that the Riemann curvature R_Y of a Finsler metric F can be expressed as the Riemann curvature $\hat{R}_Y = \hat{R}(\cdot, Y)Y$ of the induced Riemannian metric $\hat{g} := g_Y$ by a geodesic field Y. Thus their Ricci curvatures are equal in the direction of Y, $\mathrm{Ric}(Y) = \widehat{\mathrm{Ric}}(Y)$, as the trace of their Riemann curvature R_Y and $\hat{R}_Y$, respectively. For a volume form dV, one has the notion of distortion $\tau = \tau(x, y)$. The S-curvature is the rate of change of τ along a geodesic. In the direction of Y_x, $dV = e^{-f(x)} dV_{\hat{g}}$, where $f(x) = \tau(x, Y_x)$. Then $\Delta = \hat{\Delta}_f$ where $\widehat{\mathrm{Hess}}(f)$ denotes the Hessian of f with respect to $\hat{g}$. Further, $S(x, Y_x) = Y_x[\tau(\cdot, Y)] = df(Y_x)$ and $\dot{S}(x, Y_x) = Y_x[S(\cdot, Y)] = \widehat{\mathrm{Hess}}(f)(Y_x)$, Therefore

$$\mathrm{Ric}^N(x, \nabla\rho_x) = \widehat{\mathrm{Ric}}_f^N(x, \hat{\nabla}\rho_x),$$

where $\widehat{\mathrm{Ric}}_f^N$ denotes the weighted Ricci curvature of $(\hat{g}, dV = e^{-f} dV_{\hat{g}})$. Therefore estimates on $\hat{\Delta}_f \rho$ under a lower bound $\widehat{\mathrm{Ric}}_f^N \geq (N-1)H$ will be carried over to $\Delta\rho$ under a lower bound $\mathrm{Ric}^N \geq (N-1)H$. That is, the results in [8][10] give rise to estimates on the Laplacian $\Delta\rho$ under certain Ricci curvature bounds ([7]).

2. Finsler Metrics

A Finsler metric F on a manifold M is a C^∞ function on $TM \setminus \{0\}$ with the following properties:

(a) $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$.
(b) $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$, $y \neq 0$, is positive definite.

By (a) and (b), one can get

$$F(x, y_1 + y_2) \leq F(x, y_1) + F(x, y_2), \qquad y_1, y_2 \in T_xM.$$

Thus at every point $x \in M$, $F_x := F|_{T_xM}$, is a norm on T_xM. The norm F_x induces a family of inner products g_y on T_xM:

$$g_y(u, v) = g_{ij}(x, y) u^i v^j, \qquad u = u^i \frac{\partial}{\partial x^i}|_x, \ v = v^i \frac{\partial}{\partial x^i}|_x.$$

The length of a curve $c : [a, b] \to M$ is given by

$$L(c) := \int_a^b F(c(t), c'(t)) dt.$$

Locally minimizing curves with constant speed are characterized by

$$\frac{d^2 x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where

$$G^i(x, y) = \frac{1}{4} g^{il}(x, y) \Big\{ \frac{\partial g_{kl}}{\partial x^j}(x, y) + \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \Big\} y^j y^k.$$

The local functions $G^i = G^i(x, y)$ form a global vector field G on TM:

$$G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

G is called the *spray* of F.

Put

$$N^i_j := \frac{\partial G^i}{\partial y^j}, \qquad \Gamma^i_{jk} := \frac{\partial^2 G^i}{\partial y^j \partial y^k}.$$

We modify the natural local frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ by

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^i}.$$

The local dual frame $\{dx^i, \delta y^i\}$ is given by

$$\delta y^i := dy^i + N^i_j dx^j.$$

The tangent space of TM at $y \in T_xM \setminus \{0\}$ has a natural decomposition

$$T_y(TM) = H_y(TM) \oplus V_y(TM),$$

where

$$H_y(TM) := \mathrm{span}\Big\{\frac{\delta}{\delta x^i}\Big\}, \qquad V_y(TM) := \mathrm{span}\Big\{\frac{\partial}{\partial y^i}\Big\}.$$

For $X = X^i \frac{\partial}{\partial x^i} \in C^\infty(TM)$ and $y \in T_xM$, define

$$D_y X = \Big\{dX^i(y) + X^j N^i_j(x,y)\Big\} \frac{\partial}{\partial x^i}|_x.$$

D is usually called a *non-linear connection* . If $F = \sqrt{g_{ij}(x)y^i y^j}$ is Riemannian, D becomes a linear connection on TM. It is the well-known *Levi-Civita connection.*

Let $\omega^i := dx^i$, $\omega^{n+i} := \delta y^i$ and

$$\omega_j^{\ i} := \Gamma^i_{jk}(x,y)dx^k.$$

Then we get the first set of structure equations

$$d\omega^i = \omega^j \wedge \omega_j^{\ i}.$$

The local curvature forms are defined by

$$\Omega_j^{\ i} := d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i}.$$

We can express $\Omega_j^{\ i}$ as

$$\Omega_j^{\ i} := \frac{1}{2} R_j^{\ i}{}_{kl} \omega^k \wedge \omega^l - B_j^{\ i}{}_{kl} \omega^k \wedge \omega^{n+l}$$

with $R_j^{\ i}{}_{kl} + R_j^{\ i}{}_{lk} = 0$. We obtain two important curvatures: the Riemann curvature tensor $R_j^{\ i}{}_{kl}$ and the Berwald curvature tensor $B_j^{\ i}{}_{kl}$. Put

$$R^i{}_k := y^j R_j^{\ i}{}_{kl} y^l.$$

Then

$$R_j^{\ i}{}_{kl} = \frac{1}{3}\Big\{R^i{}_{k\cdot l\cdot j} - R^i{}_{l\cdot k\cdot j}\Big\}.$$

We obtain a family of linear maps $R_y : T_xM \to T_xM$,

$$R_y(u) = R^i{}_k(x,y)u^k \frac{\partial}{\partial x^i}|_x, \qquad u = u^k \frac{\partial}{\partial x^k}|_x \in T_xM.$$

It is called the *Riemann curvature.* In local coordinates, $R^i{}_k$ can be expressed by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

3. Geodesic Fields

We shall describe the Riemann curvature via geodesic fields from the Riemann-geometry point view. Let (M, F) be a Finsler manifold. A vector field Y on an open subset $U \subset M$ is called a *geodesic field* if every integral curve $c(t)$ of Y in U is a geodesic of F:

$$c'(t) = Y_{c(t)}.$$

In local coordinates, a geodesic field $Y = Y^i \frac{\partial}{\partial x^i}$ are characterized by

$$Y^j(x)\frac{\partial Y^i}{\partial x^j}(x) + 2G^i(x, Y_x) = 0. \tag{3.1}$$

Here we identify $Y_x = Y^i(x)\frac{\partial}{\partial x^i}|_x$ with $(Y^1(x), \cdots, Y^n(x))$. For any non-zero vector $y \in T_xM$, there is an open neighborhood U_x and a geodesic field Y on U_x such that $Y_x = y$. Y is called a *geodesic extension* of y. The geodesic vector field Y on U induces a Riemannian metric $\hat{g} := g_Y$ on U.

$$\hat{g}_z(u, v) := g_{Y_z}(u, v), \qquad z \in U.\ u, v \in T_zU.$$

Let $\hat{D}$ denote the Levi-Civita connection of $\hat{g}$ on U. We have the following

Lemma 3.1. *In local coordinates (x^i, y^i) in TM,*

$$\hat{G}^i(x, Y_x) = G^i(x, Y_x), . \tag{3.2}$$
$$\hat{N}^i_j(x, Y_x) = N^i_j(x, Y_x),$$

Proof: $N^i_j := \frac{\partial G^i}{\partial y^j}$ are given by

$$N^i_j = \frac{1}{2}g^{il}\Big\{\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\Big\}y^k - 2g^{il}C_{jkl}G^k. \tag{3.3}$$

Note that

$$\frac{\partial \hat{g}_{ij}}{\partial x^k}(x) = \frac{\partial g_{ij}}{\partial x^k}(x, Y_x) + 2C_{ijl}(x, Y_x)\frac{\partial Y^l}{\partial x^k}(x). \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned}
\hat{N}^i_j(x, Y_x) &= \frac{1}{2}\hat{g}^{il}(x)\Big\{\frac{\partial \hat{g}_{jl}}{\partial x^k}(x) + \frac{\partial \hat{g}_{kl}}{\partial x^j}(x) - \frac{\partial \hat{g}_{jk}}{\partial x^l}(x)\Big\}Y^k(x) \\
&= \frac{1}{2}g^{il}(x, Y_x)\Big\{\frac{\partial g_{jl}}{\partial x^k}(x, Y_x) + \frac{\partial g_{kl}}{\partial x^j}(x, Y_x) - \frac{\partial g_{jk}}{\partial x^l}(x, Y_x)\Big\}Y^k(x) \\
&\quad -2g^{il}(x, Y_x)C_{jkl}(x, Y_x)G^k(x, Y_x) \\
&= N^i_j(x, Y_x).
\end{aligned}$$

This gives (3.2). By (3.2), we obtain

$$2G^i(x, Y_x) = N^i_j(x, Y_x)Y^j(x) = \hat{N}^i_j(x, Y_x)Y^j(x) = 2\hat{G}^i(x, Y_x).$$

Thus Y also satisfies

$$Y^j(x)\frac{\partial Y^i}{\partial x^j}(x) + 2\hat{G}^i(x, Y_x) = 0.$$

Thus Y is also a geodesic field of $\hat{F}$. □

Lemma 3.2. *Let Y be a geodesic field of F on an open subset U and $\hat{g} := g_Y$. Then Y is also a geodesic field of $\hat{g}$.*

Proof: Y satisfies (3.1). By Lemma 3.1, Y satisfies

$$Y^j(x)\frac{\partial Y^i}{\partial x^j}(x) + 2\hat{G}^i(x, Y_x) = 0.$$

Thus Y is a geodesic field of $\hat{g}$ too. □

Lemma 3.3. *Let Y be a geodesic field of F on an open subset U and $\hat{g} := g_Y$. For any vector field W on U,*

$$D_Y W = \hat{D}_Y W.$$

Proof:

$$\begin{aligned} D_{Y_x} W &= \Big\{ Y_x(W^i) + W^j(x) N^i_j(x, Y_x) \Big\} \frac{\partial}{\partial x^i}|_x \\ &= \Big\{ Y_x(W^i) + W^j(x) \hat{N}^i_j(x, Y_x) \Big\} \frac{\partial}{\partial x^i}|_x = \hat{D}_{Y_x} W. \end{aligned}$$

□

One can use the non-linear connection D to define the covariant derivative $D_{\dot{c}} X(t)$ of a vector field $X = X^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ along a curve $c(t)$, $a \le t \le b$.

$$D_{\dot{c}} X(t) := \Big\{ \frac{dX^i}{dt}(t) + X^j(t) N^i_j(x(t), x'(t)) \Big\} \frac{\partial}{\partial x^i}|_{c(t)}.$$

Let $c(t), a \le t \le b$ be a geodesic in (M, F). Let

$$H : (-\varepsilon, \varepsilon) \times [a, b] \to M$$

be a geodesic variation of c, that is, $c_s(t) = H(s, t)$ is a geodesic of F for each s and $c_0(t) = c(t)$. Put

$$J(t) := \frac{\partial H}{\partial s}(0, t).$$

$J(t)$ satisfies the Jacobi equation:

$$D_{\dot{c}} D_{\dot{c}} J(t) + R_{\dot{c}}(J(t)) = 0,$$

where $R_{\dot{c}}$ is the Riemann curvature in the direction of $\dot{c}$.

We may assume that H is an embedding and $\frac{\partial H}{\partial t}(s, t)$ can be extended to a geodesic field Y in a neighborhood U of c so that

$$Y_{H(s,t)} = \frac{\partial H}{\partial t}(s, t).$$

Note that $Y_{c(t)} = c'(t)$. Since each $c_s(t) = H(s, t)$ is a geodesic of $\hat{g}$ by Lemma 3.2, $J(t)$ is a Jacobi field of $\hat{g}$. Let $\hat{R}$ denote the Riemann curvature of $\hat{g} := g_Y$. It is proved in Riemann geometry that $J(t)$ satisfies

$$\hat{D}_{\dot{c}} \hat{D}_{\dot{c}} J(t) + \hat{R}_{\dot{c}}(J(t)) = 0.$$

By Lemma 3.3, $D_{\dot{c}} = \hat{D}_{\dot{c}}$, we get

$$R_{\dot{c}}(J(t)) = \hat{R}_{\dot{c}}(J(t)).$$

We conclude that $R_y = \hat{R}_y$. We obtain the following

Lemma 3.4. *Let $y \in T_xM \setminus \{0\}$ and Y be a geodesic extension of y. Let $\hat{R}$ denote the Riemann curvature of $\hat{g} := g_Y$. Then $R_y = \hat{R}_y$. Moreover,*

$$g_y(R_y(u), v) = g_y(u, R_y(v)), \qquad u, v \in T_xM.$$

Proof: It has been proved in Riemannian geometry that

$$\hat{g}_x(\hat{R}_y(u), v) = \hat{g}_x(u, \hat{R}_y(v)), \qquad u, v \in T_xM.$$

Note that $\hat{g}_x = g_y$ and $\hat{R}_y = R_y$. This completes the proof. □

The Ricci curvature $\mathrm{Ric}(x, y)$ is the trace of the Riemann curvature $R_y : T_xM \to T_xM$.

$$\mathrm{Ric}(x, y) := \mathit{trace}(R_y) = \sum_{i=1}^n R^i{}_i(x, y).$$

We have the following

Lemma 3.5. *Let $y \in T_xM \setminus \{0\}$ and Y be a geodesic extension of y. Let $\widehat{\mathrm{Ric}}$ denote the Ricci curvature of $\hat{g} := g_Y$. Then*

$$\mathrm{Ric}(x, y) = \widehat{\mathrm{Ric}}(x, y).$$

4. Volume Form and S-curvature

Let (M, F) be a Finsler manifold. The Finsler metric F determines a distance function d_F. The distance function d_F determines the Hausdorff measure μ_F. H. Busemann finds a volume form dV_F for the Hausdorff measure

$$\mu_F(U) = \int_U dV_F,$$

In local coordinates (x^i), $dV_F = \sigma_F(x)dx^1 \cdots dx^n$, is given by

$$\sigma_F(x) = \frac{\mathrm{Vol}(B^n(1))}{\mathrm{Vol}\{(y^i) \in R^n \mid F(x,y) < 1\}}.$$

Example 4.1. Let $F(x,y) = \sqrt{g_{ij}(x)y^i y^j}$ be a Riemannian metric. The Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is given by

$$\sigma_F(x) = \sqrt{\det(g_{ij}(x))}.$$

Example 4.2. Let $F = \alpha(x,y) + \beta(x,y)$ be a Randers metric on M where

$$\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}, \qquad \beta(x,y) = b_i(x)y^i,$$

with

$$b(x) := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1,$$

where $(a^{ij}(x)) = (a_{ij}(x))^{-1}$. The Busemann-Hausdorff volume $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is given by

$$\sigma_F(x) = \Big(1 - b(x)^2\Big)^{\frac{n+1}{2}} \sigma_\alpha(x),$$

where $\sigma_\alpha(x) = \sqrt{\det(a_{ij}(x))}$.

A Randers metric $F = \alpha + \beta$ can be also expressed in the following navigation form:

$$F = \frac{\sqrt{(1-\bar{b}^2)\bar{\alpha}^2 + \bar{\beta}^2}}{1-\bar{b}^2} - \frac{\bar{\beta}}{1-\bar{b}^2}, \tag{4.1}$$

where $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x)y^i y^j}$ is a Riemannian metric and $\bar{\beta} = \bar{b}_i(x)y^i$ is a 1-form with $\bar{b}(x) := \|\bar{\beta}_x\|_{\bar{\alpha}} < 1$. We have

$$dV_F = dV_{\bar{\alpha}}.$$

Randers metrics in the form (4.1) are called *general* (α, β)*-metrics*. However, for other general (α, β)-metrics, it is impossible to find an explicit formula for the Busemann-Hausdorff volume form.

Let dV be a volume form on (M, F). In local coordinates

$$g_{ij}(x,y) = \frac{1}{2}[F^2]_{y^i y^j}(xy), \qquad dV = \sigma(x)dx^1 \cdots dx^n.$$

Then the following quantity is well-defined

$$\tau(x,y) := \ln \frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)}.$$

$\tau = \tau(x,y)$ is called the *distortion* of (F, dV). The vertical covariant derivative is the *mean Cartan torsion*:

$$I_i(x,y) = \tau_{y^i}(x,y) = g^{jk}(x,y)C_{ijk}(x,y).$$

Brickell's Theorem says that for a regular Finsler metric F, $I = 0$ if and only if F is Riemannian ([1]).

The derivative of the distortion along a geodesic is the so-called S-curvature

$$S(x,y) := \frac{d}{dt}\Big[\tau(c(t), c'(t))\Big]_{|t=0}$$

where $c(t)$ is the geodesic with $c(0) = x$ and $c'(0) = y$. In local coordinates, if $G^i = G^i(x, y)$ denote the spray coefficients of F and $dV = \sigma(x)dx^1 \cdots dx^n$, then

$$S(x, y) = \frac{\partial G^m}{\partial y^m}(x, y) - y^m \frac{\partial}{\partial}\Big[\ln \sigma(x)\Big].$$

Thus the S-curvature is also defined for a spray G and a volume form dV.

Let Y be a geodesic field on an open subset U and $\hat{g} := g_Y$. Observe that

$$\tau(x, Y_x) = \ln \frac{\sqrt{\det(g_{ij}(x, Y_x))}}{\sigma(x)} = \ln \frac{\sqrt{\det(\hat{g}_{ij}(x))}}{\sigma(x)}.$$

Let

$$f(x) := \tau(x, Y_x), \qquad x \in U.$$

Then

$$dV = e^{-f(x)} dV_{\hat{g}}.$$

The S-curvature of (F, dV) is given by

$$S(x, Y_x) = Y_x[\tau(\cdot, Y)] = df_x(Y_x).$$

5. The Gradient and Laplacian

For a Finsler metric F on a manifold M, the dual co-Finsler metric F^* is a function on T^*M, defined by

$$F^*(x, \eta) := \sup_{y \in T_xM} \frac{\eta(y)}{F(x, y)}.$$

Conversely, F can be viewed as the dual metric to F^* by the following identity:

$$F(x, y) = \sup_{\eta \in T_x^*M} \frac{\eta(y)}{F^*(x, \eta)}.$$

The Lagrange $\mathcal{L}_x : T_xM \to T_x^*M$ is defined by

$$\mathcal{L}_x(y) := g_y(y, \cdot).$$

The Lagrange map $\mathcal{L}_x$ is positively homogeneous in $y \in T_xM$, that is, $\mathcal{L}_x(\lambda y) = \lambda \mathcal{L}_x(y)$, $\forall \lambda > 0$. Further, $\mathcal{L}_x : T_xM \setminus \{0\} \to T_x^*M \setminus \{0\}$ is a diffeomorphism with

$$F^*(x, \mathcal{L}_x(y)) = F(x, y).$$

Definition 5.1. *Let f be a C^∞ function on M and $x \in M$. If $df_x \neq 0$, set*

$$\nabla f_x := \mathcal{L}_x^{-1}(df_x).$$

If $df_x = 0$, set

$$\nabla f_x = 0.$$

∇f_x is called the gradient of f at x.

From the definition of ∇f_x, we have

$$df_x = \mathcal{L}_x(\nabla f_x) = g_{\nabla f_x}(\nabla f_x, \cdot).$$

∇f is C^∞ on the open set $\{df_x \neq 0\}$. Let $g^{*ij}(x, \eta) := \frac{1}{2}\frac{\partial^2}{\partial \eta^i \eta^j}[F^{*2}](x, \eta)$. Then

$$\nabla f_x = \nabla^i f(x) \frac{\partial}{\partial x^i}|_x = g^{*ij}(x, df) \frac{\partial f}{\partial x^j}(x) \frac{\partial}{\partial x^i}|_x.$$

Let f be a C^∞ function on an open subset $U \subset M$ and $N_t := f^{-1}(t) \subset U$. Suppose that $df_x \neq 0$ at some point $x \in N_t$, then N_t is a hypersurface in a neighborhood of x. We have

$$g_{\nabla f_x}(\nabla f_x, v) = df_x(v) = 0, \qquad \forall v \in T_xN_t.$$

Namely, ∇f_x is *perpendicular* to N_t at x with respect to $g_{\nabla f_x}$.

Assume that $df_x \neq 0$ on an open set U. Let $\hat{g} := g_{\nabla f}$ and $\hat{\nabla} f$ denote the gradient of f with respect to $\hat{g}$. We have the following

Lemma 5.2.

$$\nabla f = \hat{\nabla} f. \tag{5.1}$$

Further

$$F(x, \nabla f_x) = \sqrt{\hat{g}_x(\hat{\nabla} f_x, \hat{\nabla} f_x)} = \|\hat{\nabla} f_x\|_{\hat{g}}.$$

Proof: For any tangent vector $v \in T_xM$,

$$\hat{g}(\hat{\nabla} f, v) = df(v) = g_{\nabla f}(\nabla f, v) = \hat{g}(\nabla f, v).$$

This implies (5.1).

Observe that

$$\begin{aligned} F(x, \nabla f_x)^2 &= g_{\nabla f_x}(\nabla f_x, \nabla f_x) \\ &= \hat{g}_x(\nabla f_x, \nabla f_x) \\ &= \hat{g}_x(\hat{\nabla} f_x, \hat{\nabla} f_x). \end{aligned}$$

□

For a vector $y \in T_xM$, define the *Hessian* Hess(f) of f at x by

$$\mathrm{Hess}(f)(y) := \frac{d^2}{dt^2}\Big[f \circ c_y(t)\Big]|_{t=0},$$

where $c_y(t)$ is the geodesic with $c_y(0) = x$ and $c'_y(0) = y$. Hess$(f)(\lambda y) = \lambda^2 Hess(f)(y)$, $\lambda > 0$. But Hess$(f)(y)$ is not quadratic in $y \in T_xM$. In local coordinates

$$\mathrm{Hess}(f)(y) = \frac{\partial^2 f}{\partial x^i \partial x^j}(x) y^i y^j - 2G^i(x, y)\frac{\partial f}{\partial x^i}(x).$$

It is easy to prove the following

Lemma 5.3. *Let f be a function on M,*

$$\mathrm{Hess}(f)(y) = y[Y(f)].$$

where Y is a geodesic extension of y.

Let $dV = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ be a volume form on M and $X = X^i \frac{\partial}{\partial x^i}$ a vector field on M. The *divergence* of X with respect to dV is given by

$$\mathrm{div}(X) = \frac{1}{\sigma(x)} \frac{\partial}{\partial x^i}\Big[\sigma(x) X^i(x)\Big].$$

On a Finsler metric measure manifold (M, F, dV), the Laplacian of a C^∞ function f on M is defined by

$$\Delta f := \mathrm{div}(\nabla f).$$

Δf is well-defined in a usual sense on $U := \{x \in M \mid df_x \neq 0\}$. In local coordinates

$$\Delta f = \frac{\partial}{\partial x^i}\Big(\nabla^i f(x)\Big) + \nabla^i f(x) \frac{\partial}{\partial x^i}\Big[\ln \sigma(x)\Big].$$

However, $\nabla \rho$ is not C^∞ at a point where $df = 0$. Thus Δf is defined on the whole manifold in a weak sense.

6. Distance Functions

Let A be a closed subset in a Finsler manifold (M, F). Let

$$\rho_+(x) := d(A, x), \qquad \rho_-(x) := -d(x, A).$$

ρ_+ and ρ_- are locally Lipschitz functions. Thus they are differentiable almost everywhere. We have the following

Lemma 6.1. *Let $\rho(x) = \rho_+(x)$ or $\rho_-(x)$. Assume that f is C^∞ on an open subset $U \subset M$. Then*

$$F^*(x, d\rho_x) = F(x, \nabla\rho_x) = 1, \qquad x \in U.$$

Therefore we make the following

Definition 6.2. *A Lipschitz function f on a Finsler manifold (M, F) is called a distance function if the following identity holds almost everywhere on M*

$$F(x, \nabla f_x) = 1.$$

Let f be a C^∞ function on an open subset $U \subset M$ with $df_x \neq 0$, $\forall x \in U$. Let $\hat{g} := g_{\nabla f}$ be the induced Riemannian metric on U. By Lemma 5.2, we have

$$\nabla f = \hat{\nabla} f.$$

Further, $F(x, \nabla f_x) = \sqrt{\hat{g}(x, \hat{\nabla} f_x)}$. Thus $F(x, , \nabla f_x) = 1$ if and only of $\hat{g}(x, \hat{\nabla} f_x) = 1$. That is, f is a distance function of F if and only if f is a distance function of $\hat{g}$.

Proposition 6.3. *Let $\rho = \rho(x)$ be a C^∞ distance function on $U \subset M$. Let $\hat{g} := g_{\nabla\rho}$. Then $\nabla\rho = \hat{\nabla}\rho$ is a geodesic field of F and $\hat{g}$.*

Proof: ρ is a distance function of F. We have

$$g_{\nabla\rho}(\nabla\rho, \nabla\rho) = 1.$$

Let V be a vector field on U such that $V \perp \nabla\rho$ with respect to $g_{\nabla\rho}$ and $[V, \nabla\rho] = 0$. One can choose V in the following way. Take a variation $H : (-\varepsilon, \varepsilon) \times (a, b) \to M$ such that

$$\nabla\rho|_{H(s,t)} = \frac{\partial H}{\partial t}(s, t), \qquad V_{H(s,t)} = \frac{\partial H}{\partial s}(s, t).$$

We can make $V \perp \nabla\rho$ with respect to $g_{\nabla\rho}$. Then

$$0 = V g_{\nabla\rho}(\nabla\rho, \nabla\rho) = 2g_{\nabla\rho}(\nabla\rho, D_{\nabla\rho}V) = -2g_{\nabla\rho}(D_{\nabla\rho}\nabla\rho, V).$$

$$0 = \nabla\rho[g_{\nabla\rho}(\nabla\rho, \nabla\rho)] = 2g_{\nabla\rho}(D_{\nabla\rho}\nabla\rho, \nabla\rho).$$

Thus $D_{\nabla\rho}\nabla\rho = 0$. This implies that $\nabla\rho$ is a geodesic field of F. By a similar argument, one can show that $\hat{\nabla}\rho$ is a geodesic field of $\hat{g}$. □

7. Mean Curvature

Let $i : N \to M$ be an embedded hypersurface in a Finsler manifold (M, F) and dV be a volume form on M. Let $\mathbf{n}$ be a normal vector to N at $x \in N$,

$$g_{\mathbf{n}}(\mathbf{n}, v) = 0, \qquad \forall v \in T_x N.$$

Let $e_1 = \mathbf{n}, e_a, 2 \leq a \leq n$ be an orthonormal basis for $(T_x M, g_{\mathbf{n}})$. Let $\{\theta^i\}_{i=1}^n$ be the dual basis for T^*M. Then $\theta^1(v) = 0$ for all $v \in T_x N$ and $\{i^*\theta^a\}_{a=2}^n$ is a basis for T^*N.

Let $dV_x = \sigma(x)\theta^1 \wedge \cdots \wedge \theta^n$ at $x \in N \subset M$. The induced volume form dA_x at $x \in N$ is given by

$$dA_x = \sigma(x) i^*\theta^2 \wedge \cdots \wedge i^*\theta^n.$$

Locally, N can be viewed as a level surface of a distance function ρ so that $\mathbf{n} = \nabla\rho|_N$ is a normal vector to N. We may assume that N is contained in an open subset $U \subset M$ and it divides U into two disconnected open subsets U_- and U_+. Thus $U = U_- \cup N \cup U_+$. Define $\rho : U \to R$ by

$$\rho(x) := d(N, x), \qquad x \in U_+$$

$$\rho(x) := -d(x, N), \qquad x \in U_-.$$

ρ is a C^∞ distance function on U with $N = \rho^{-1}(0)$. Then ρ has the required property.

For $x \in N$, let $c_x(t)$ denote the integral curve of $\nabla\rho$ with $c_x(0) = x$. For a mall $\varepsilon > 0$, let $N_\varepsilon := \rho^{-1}(\varepsilon)$ and define $\phi_\varepsilon : N \to N_\varepsilon$ by

$$\phi_\varepsilon(x) = c_x(\varepsilon). \tag{7.1}$$

Let dA_ε denote the induced volume form on N_ε. Let $dA = dA_0$. Set

$$\phi_\varepsilon^* dA_\varepsilon = \Theta_\varepsilon(x) dA.$$

Note that $\Theta_0(x) = 1$.

Definition 7.1. *The mean curvature of N at $x \in N$ is defined by*

$$m(x) := \frac{d}{d\varepsilon}\Big[\ln \Theta_\varepsilon(x)\Big]|_{\varepsilon=0}.$$

In the following, we are going to show that the Laplacian of a distance function is the mean curvature of the level surface of the distance function.

Let ρ be a C^∞ distance function on an open subset $U \subset M$. Let $N := N_0 := \rho^{-1}(0)$. There is a local coordinate system (t, x^a) such that (x^a) is a local system on N and

$$\rho(t, x^a) = t.$$

We have

$$\nabla\rho = \frac{\partial}{\partial t}.$$

Let $x^1 := t$ and $\nabla\rho = \nabla^i\rho\frac{\partial}{\partial x^i}$. Then

$$\nabla^1\rho = 1, \qquad \nabla^a\rho = 0 \ \ (a = 2, \cdots, n).$$

Let

$$\theta^1 = dt, \qquad \theta^a = dx^a.$$

Put $dV = \sigma(t, x^a)dt \wedge dx^2 \wedge \cdots \wedge dx^n$. Then the induced volume form dA on N_t is given by

$$dA|_{N_t} = \sigma(t, x^a)dx^2 \wedge \cdots \wedge dx^n.$$

The Laplacian $\Delta\rho$ of ρ can be expressed as

$$\begin{aligned}\Delta\rho &= \frac{\partial}{\partial x^i}(\nabla^i\rho) + \nabla^i\rho\frac{\partial}{\partial x^i}(\ln\sigma)\\ &= \nabla^1\rho\frac{\partial}{\partial x^1}(\ln\sigma).\end{aligned}$$

We obtain

$$\Delta\rho(x) = \frac{\partial}{\partial t}\Big[\ln\sigma(t, x^a)\Big]|_{t=\rho(x)}.$$

The map $\phi_\varepsilon : N_t \to N_{t+\varepsilon}$ is given by

$$\phi_\varepsilon(t, x^a) = (t + \varepsilon, x^a).$$

The pull-back volume form $\phi_\varepsilon^* dA|_{N_{t+\varepsilon}} = \Theta_\varepsilon(t, x^a)dA_{N_t}$ is given by

$$\phi_\varepsilon^* dA|_{N_{t+\varepsilon}} = \sigma(t+\varepsilon, x^a)dx^2 \wedge \cdots \wedge dx^n = \frac{\sigma(t+\varepsilon, x^a)}{\sigma(t, x^a)}dA_{N_t}.$$

That is

$$\Theta_\varepsilon(t, x^a) = \frac{\sigma(t+\varepsilon, x^a)}{\sigma(t, x^a)}.$$

Therefore the mean curvature $m(x)$ at $x \in N_t$ is given by

$$m(x) = \frac{d}{d\varepsilon}\Big[\frac{\sigma(t+\varepsilon, x^a)}{\sigma(t, x^a)}\Big]|_{\varepsilon=0} = \frac{\partial}{\partial t}\Big[\ln\sigma(t, x^a)\Big].$$

We have proven the following

Lemma 7.2. *The Laplacian of a distance function ρ on U is the mean curvature of the level surface.*

$$\Delta\rho|_x = m(x), \qquad x \in N := \rho^{-1}(0).$$

8. Volume of Geodesic Balls

In this section, we are going to express the volume of metric balls an integral over the unit tangent sphere at a point. Let (M, F) be a positively complete manifold. Let $p \in M$ and $\exp_p : T_pM \to M$ be the exponential map so that $\exp_p(T_pM) = M$. The exponential map $\exp_p$ is C^∞ on $T_pM \setminus \{0\}$ and C^1 at $0 \in T_pM$ such that

$$(\exp_p)_*|_0 : T_0(T_pM) \equiv T_pM \to T_pM$$

is an identity map.

For a unit vector $y \in S_pM$, the conjugate value c_y of y is the first zero r of a Jacobi filed $J(t)$ along $c(t) = \exp_p(ty)$, equivalently, the smallest positive number $r > 0$ such that $d(\exp_p)|_{ry} : T_{ry}(T_pM) \to T_{\exp_p(ry)}M$ is singular.

For a vector $y \in S_pM$, the injectivity value i_y of y is the largest possible value r such that $c_y|_{[0,r]}$ is a minimizing geodesic. The cut-domain Ω_p is defined by

$$\Omega_p := \Big\{ \exp_p(ty) \mid y \in S_pM, 0 \le t < t_y \Big\}.$$

Ω_p is a star-shaped open domain in M. Further the cut-locus $Cut(p) := M \setminus \Omega_p$ has zero measure. Let

$$T\Omega_p := \{ty \in T_pM, \mid 0 < t < i_y, \; y \in S_pM\}.$$

$\exp_p$ is a diffeomorphism from $T\Omega_p$ to an open subset $\Omega_p = \exp_p(T\Omega_p)$. Let dV_p denote the restriction of dV on T_pM. It induces a volume form dA_p on the unit tangent sphere S_pM. Define a map $\phi : [0, \infty) \times S_pM \to M$ by

$$\phi(t, y) = \exp_p(ty).$$

Let $S^r_pM := \{y \in S_pM \mid r < i_y\}$ and $\tilde{S}(p, r) := S(p, r) \cap \Omega_p$. Then $\varphi_t = \phi(r, \cdot) : S^t_pM \to \tilde{S}(p, r)$ is a diffeomorphism.

Let

$$\varphi_r^* dA_{\tilde{S}(p,r)} = \eta_r(p, y) dA_p, \qquad y \in S^r_pM.$$

Recall the map $\phi_\varepsilon : S(p, r) \to S(p, r + \varepsilon)$ defined in (7.1) and put

$$\phi_\varepsilon^* dA|_{S(p,r+\varepsilon)} = \Theta_\varepsilon(y) dA_{|S(p,r)}.$$

It is easy to see that and $x = \exp_p(ry)$,

$$\Theta_\varepsilon(x) = \frac{\eta_{r+\varepsilon}(p, y)}{\eta_r(p, y)}.$$

Then the mean curvature at x is given by

$$m(x) = \frac{d}{d\varepsilon} \ln \Theta_\varepsilon(x)|_{\varepsilon=0} = \frac{d}{dr} \ln \eta_r(p, y).$$

The volume of $\tilde{S}(p, r)$ and $B(p, R) \setminus B(p, r)$ can be expressed as an integral of $\eta_r(p, y)$.

$$\mathrm{Vol}(\tilde{S}(p, r)) = \int_{\tilde{S}(p,r)} dA_{\tilde{S}(p,r)} = \int_{S^r_pM} \varphi_r^* dA_{\tilde{S}(p,r)} = \int_{S^r_pM} \eta_r(p, y) dA_p.$$

$$\mathrm{Vol}(B(p, R) \setminus B(p, r)) = \int_r^R \mathrm{Vol}(\tilde{S}(p, t)) dt = \int_r^R \int_{S^t_pM} \eta_t(p, y) dA_p dt.$$

Therefore, estimates on the mean curvature $m(x)$ along a geodesic $c_y(t) = \exp_p(ty)$ will gives estimates on $\mathrm{Vol}(\tilde{S}(p, r))$ and then on $\mathrm{Vol}(B(p, R) \setminus B(p, r))$.

9. Curvature-free comparison theorems

Let $0 \le \rho_o < t_o \le +\infty$ and a C^∞ function

$$\chi : (\rho_o, t_o) \to (0, +\infty), \tag{9.1}$$

if $\rho_o = 0$ then $\lim_{t\to 0^+} \chi(t) = 0$ and if $t_o < +\infty$, then $\lim_{t \to t_o^-} \chi(t) = 0$.

A typical example is tat $\chi(t) = e^{\delta t}[S_H(t)]^{n-1}$, $0 = \rho_o < t < t_o$, where $s_H(t) = f(t)$ is the unique solution to the following equation:

$$f''(t) + Hf(t) = 0, \qquad f(0) = 0, \; f'(0) = 1.$$

$t_o := \pi/\sqrt{H}$ if $H > 0$ and $t_o = +\infty$ if $H \le 0$.

Lemma 9.1. *Let χ be a function in (9.1). Assume that for $x \in [B(p,t_o) \setminus B(p,\rho_o)] \cap \Omega_p$*

$$\Delta \rho_x \leq \frac{d}{dt}\Big[\ln \chi(t)\Big]|_{t=\rho(x)}.$$

Then the injectivity value $i_y \leq i_o$ for any $y \in S_pM$.

Proof: By assumption we have

$$\frac{d}{dt}\ln \eta_t(p,y) \leq \frac{d}{dt}\ln \chi(t), \qquad 0 < t < \min(t_o, i_y).$$

$$\frac{d}{dt}\ln \frac{\eta_t(p,y)}{\chi(t)} \leq 0, \qquad 0 < t < \min(t_o, i_y).$$

For a sufficiently small $\varepsilon > 0$,

$$\eta_t(p,y) \leq \chi(t)\frac{\eta_{\rho_o+\varepsilon}(p,y)}{\chi(\rho_o+\varepsilon)}, \qquad 0 < \varepsilon < t < \min(t_o, i_y).$$

We claim that $i_y \leq t_o$. Suppose it is not true, i.e., $t_o < i_y$. Then

$$0 < \eta_t(p,y) \leq \chi(t)\frac{\eta_{\rho_o+\varepsilon}(p,y)}{\chi(\rho_o+\varepsilon)}, \qquad 0 < t < i_o < i_y.$$

Letting $t \to i_o^-$, one gets

$$\eta_{t_o}(p,y) \leq 0.$$

This is impossible. Therefore $i_y \leq i_o$. □

Therefore we make the following

Assumption X: Let χ be a function defined in (9.1), For a point $p \in M$, $d_p := \sup_{x\in M} d(p,x) \leq t_o$ and

$$\Delta \rho(x) = m(x) \leq \frac{d}{dr}\Big[\ln \chi(r)\Big]|_{r=\rho(x)}, \qquad x \in \Omega_p \setminus B(p,\rho_o).$$

Under **Assumption X**, one can easily see that along any minimizing geodesic $c_y(t) = \exp_p(ty)$, $\rho_o < t < i_y$,

$$\frac{d}{dr}\Big[\ln \frac{\eta_r(p,y)}{\chi(r)}\Big] \leq 0.$$

Thus for any $\rho_o < \tau < t < t_o$,

$$\eta_t(p,y)\chi(\tau) \leq \eta_\tau(p,y)\chi(t).$$

Integrating the above inequality over S_p^tM, we get

$$\mathrm{Vol}(\tilde{S}(p,t))\chi(\tau) \leq \mathrm{Vol}(\tilde{S}(p,\tau))\chi(t).$$

Integrating the above identity with respect to τ over $[\rho_o, \rho]$, we get

$$\mathrm{Vol}(\tilde{S}(p,t))\int_{\rho_o}^{\rho}\chi(\tau)d\tau \leq \mathrm{Vol}(B(p,\rho) \setminus B(p,\rho_o))\chi(t)$$

Integrating the above inequality with respect to t over $[\rho, r]$, we get

$$\mathrm{Vol}(B(p,r) \setminus B(p,\rho))\int_{\rho_o}^{\rho}\chi(\tau)d\tau \leq \mathrm{Vol}(B(p,\rho) \setminus B(p,\rho_o)\int_{\rho}^{r}\chi(t)dt$$

Adding $\mathrm{Vol}(B(p,\rho) \setminus B(p,\rho_o))\int_{\rho_o}^{\rho}\chi(\tau)d\tau$ to both sides yields

$$\mathrm{Vol}(B(p,r) \setminus B(p,\rho_o))\int_{\rho_o}^{\rho}\chi(\tau)d\tau \leq \mathrm{Vol}(B(p,t) \setminus B(p,\rho_o))\int_{\rho_o}^{r}\chi(t)dt$$

We have the following

Proposition 9.2. *([2]) Let $\chi : (\rho_o, t_o) \to (0, +\infty)$. Assume that for a point $p \in M$, Assumption X holds. Then for $\rho_o < \rho < r < t_o$,*

$$\frac{\mathrm{Vol}(\tilde{S}(p,r))}{\mathrm{Vol}(B(p,\rho_o)) \setminus B(p,r))} \leq \frac{\chi(r)}{\int_{\rho_o}^{r} \chi(s)ds}.$$

$$\frac{\mathrm{Vol}(B(p,r) \setminus B(p,\rho_o))}{\mathrm{Vol}(B(p,\rho)) \setminus B(p,\rho_o))} \leq \frac{\int_{\rho_o}^{r} \chi(t)dt}{\int_{\rho_o}^{\rho} \chi(s)ds}$$

For a unit vector $y \in S_pM$, the mean curvature $m(r) := m|_{c_y(r)}$ of $S(p,r)$ at $c_y(r)$ has the following expansion

$$m(r) = \frac{n-1}{r} - S(x,y) - \frac{1}{3}\Big[\mathrm{Ric}_p(y) + 3\dot{S}(x,y)\Big]r + o(r).$$

This is given in Proposition 14.4.5 in [9]. This gives

$$\eta_r(p,y) = r^{n-1}\Big\{1 - S(p,y)r + O(r^2)\Big\}.$$

Thus

$$\begin{aligned}\mathrm{Vol}(S(p,r)) &= \phi(p)\mathrm{Vol}(S^{-1}(1))r^{n-1}\Big\{1 - s(p)r + O(r^2)\Big\}\\ \mathrm{Vol}(B(p,r)) &= \phi(p)\mathrm{Vol}(B^n(1))r^n\Big\{1 - \frac{n}{n+1}s(p)r + O(r^2)\Big\}.\end{aligned}$$

where $dV = \phi(x)dV_{BH}$ and $s(p) := \frac{1}{\mathrm{Vol}(S^{n-1}(1))}\int_{S_pM} S(p,y)dA_p$.

10. Weighted Laplacian

Let (M, F, dV) be a Finsler metric measure manifold. Let ρ be a C^∞ distance function on $U \subset M$. Let $\hat{g} := g_{\nabla\rho}$. τ and $\hat{\tau}$ denote the distortion of (F, dV) and $(\hat{g}, dV)$, respectively. Let $f(x) = \tau(x, \nabla\rho_x)$ and $\hat{f}(x) := \hat{\tau}(x, \hat{\nabla}\rho_x)$. We have

$$f(x) = \ln\frac{\sqrt{\det(g_{ij}(x,\nabla\rho))}}{\sigma(x)} = \ln\frac{\sqrt{\det(\hat{g}_{ij}(x))}}{\sigma(x)} = \hat{f}(x).$$

Thus

$$dV = e^{-f(x)}dV_{\hat{g}}.$$

Lemma 10.1. *The S-curvature and its dot derivative of S in the direction $\nabla\rho$ are given by*

$$S(x,\nabla\rho_x) = df(\hat{\nabla}\rho_x), \qquad \dot{S}(x,\nabla\rho_x) = \widehat{\mathrm{Hess}}(f)(\hat{\nabla}\rho_x).$$

where $f(x) := \tau(x, \nabla\rho_x)$.

Proof: It follows that

$$S(x,\nabla\rho_x) = \nabla\rho_x[\tau(\cdot,\nabla\rho)] = \hat{\nabla}\rho_x(f) = df(\hat{\nabla}\rho_x).$$

$$\dot{S}(x,\nabla\rho_x) = \nabla\rho_x[S(\cdot,\nabla\rho)] = \hat{\nabla}\rho_x[df(\hat{\nabla}\rho)] = \widehat{\mathrm{Hess}}(f)(\hat{\nabla}\rho_x).$$

□

We now study the Laplacian $\Delta\rho = \div(\nabla\rho)$ of a distance function $\rho = \rho(x)$ with respect to (F, dV). Let $\hat{\Delta}_f\rho$ denote the Laplacian of ρ with respect to $(\hat{g}, dV = e^{-f}dV_{\hat{g}})$, where $f(x) := \tau(x, \nabla\rho_x)$.

Lemma 10.2.

$$\Delta\rho = \hat{\Delta}_f\rho.$$

Proof: Let $dV = \sigma(x)dx^1\cdots dx^n$ and $dV_{\hat{g}} = \hat{\sigma}(x)dx^1\cdots dx^n$. Then

$$f(x) = \tau(x,\nabla\rho_x) = \ln\frac{\hat{\sigma}(x)}{\sigma(x)}.$$

$$\begin{aligned}
\Delta\rho &= \frac{\partial}{\partial x^i}(\nabla^i\rho) + \nabla^i\rho\frac{\partial}{\partial x^i}\ln\sigma \\
&= \frac{\partial}{\partial x^i}(\hat{\nabla}^i\rho) + \hat{\nabla}^i\rho\frac{\partial}{\partial x^i}\ln\sigma \\
&= \frac{\partial}{\partial x^i}(\hat{\nabla}^i\rho) + \hat{\nabla}^i\rho\Big[\frac{\partial}{\partial x^i}\ln\hat{\sigma} - \frac{\partial}{\partial x^i}\ln\frac{\hat{\sigma}}{\sigma}\Big] \\
&= \hat{\Delta}\rho - \hat{\nabla}^i\rho\frac{\partial}{\partial x^i}\ln\frac{\hat{\sigma}}{\sigma} \\
&= \hat{\Delta}\rho - \hat{\nabla}\rho(f) \\
&= \hat{\Delta}\rho - df(\hat{\nabla}\rho) = \hat{\Delta}_f\rho.
\end{aligned}$$

□

11. Weighted Ricci Curvature

Let $F = F(x, y)$ be a Finsler metric and $dV = \sigma dx^1 \cdots dx^n$ be a volume form on an n-manifold M. Let $\mathrm{Ric} = \mathrm{Ric}_F$ denote the Ricci curvature of F and $S = S_{(F,dV)}$ denote the S-curvature of (F, dV). The weighted Ricci curvature is defined by

$$\mathrm{Ric}^N := \mathrm{Ric} + \dot{S} - \frac{1}{N-n}S^2.$$

$$\mathrm{Ric}^\infty := \mathrm{Ric} + \mathrm{Hess}(f).$$

If F is Riemannian, $dV = e^{-f}dV_F$. Then

$$S = df.$$

Thus

$$\mathrm{Ric}^N = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{N-n}(df)^2.$$

$$\mathrm{Ric}^\infty = \mathrm{Ric} + \mathrm{Hess}(f).$$

This is the well-known weighted Ricci curvature in Riemannian geometry. We are going to show that the weighted Ricci curvature of (F, dV) can be expressed as the weighted Ricci curvature of$(\hat{g}, dV)$ in the direction of Y where $\hat{g} = g_Y$ is the induced Riemannian metric induced by a geodesic field Y on an open subset.

Let Y be a C^∞ geodesic field on an open subset $U \subset M$ and $\hat{g} = g_Y$. Let

$$dV := e^{-f}dV_{\hat{g}}.$$

where f is given by

$$f(x) = \ln\frac{\sqrt{\det(\hat{g}_{ij}(x))}}{\sigma(x)} = \ln\frac{\sqrt{\det(g_{ij}(x, Y_x)}}{\sigma(x)} = \tau(x, Y_x).$$

Here $\tau = \tau(x, y)$ is the distortion of F at x.

By the definition of the S-curvature, we have

$$S(x, y) = y[\tau(\cdot, Y)] = df(y).$$

$$\dot{S}(x, y) = y[S(\cdot, Y)] = y[Y(f)] = \mathrm{Hess}(f)(y) = \widehat{\mathrm{Hess}}(f)(y).$$

That is, for $y = Y_x \in T_xM$,

$$S(x, y) = df(y), \qquad \dot{S}(x, y) = \widehat{\mathrm{Hess}}(f)(y).$$

By Lemma 3.5,

$$\mathrm{Ric}(x, Y_x) = \widehat{\mathrm{Ric}}(x, Y_x).$$

Then in the direction of Y_x

$$\mathrm{Ric}(x, Y_x) + \dot{S}(x, Y_x) - \frac{1}{N-n}S(x, Y_x)^2 = \widehat{\mathrm{Ric}}(Y_x) + \widehat{\mathrm{Hess}}(f)(Y_x) - \frac{1}{N-n}[df(Y_x)]^2.$$

This proves the following

Lemma 11.1. *Let Y be a geodesic field on an open subset U and $\hat{g} = g_Y$. Put $dV = e^{-f}dV_{\hat{g}}$. Then*

$$\mathrm{Ric}^N(x, Y_x) = \widehat{\mathrm{Ric}}{}_f^N(x, Y_x).$$

12. Comparison Theorems in Riemannian geometry

Let $(M, g, dV = e^{-f}dV_g)$ be a Riemannian metric measure manifold. Let Δ denote the Laplacian with respect to g and Δ_f denote the Laplacian with respect to (g, dV), i.e., for a C^∞ function u on M,

$$\Delta_f u = \mathrm{div}_{dV}(\nabla u) = \Delta u - df(\nabla u).$$

We have the following Bochner formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ric}(\nabla u) + g(\nabla u, \nabla(\Delta u)). \tag{12.1}$$

Using

$$\frac{1}{2}df(\nabla(|\nabla u|^2)) = g(\nabla u, \nabla(df(\nabla u))) - |\mathrm{Hess}(f)(\nabla u)|^2$$

one gets from (12.1) that

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ric}_f^N(\nabla u) + g(\nabla u, \nabla(\Delta_f u)) + \frac{1}{N-n}df(\nabla u)^2. \tag{12.2}$$

Let

$$\mathrm{Ric}_f^\infty := \mathrm{Ric} + \mathrm{Hess}(f).$$

$$\mathrm{Ric}_f^N := \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{N-n}(df)^2$$

Let $\rho(x) := d(p, x)$ be the distance function from a point $p \in M$ so that $\|\nabla\rho\| = 1$. Clearly

$$\mathrm{Hess}(\nabla\rho) = 0.$$

Thus

$$(\Delta\rho)^2 \leq (n-1)\mathrm{Hess}(\rho).$$

Letting $u = \rho(x)$ in (12.1), we obtain

$$0 \geq \frac{(\Delta\rho)^2}{n-1} + \mathrm{Ric}(\nabla\rho) + \nabla\rho(\Delta\rho). \tag{12.3}$$

Letting $u = \rho(x)$ in (12.2) , we obtain

$$0 \geq \frac{(\Delta\rho)^2}{n-1} + \mathrm{Ric}_f^N(\nabla\rho) + \nabla\rho(\Delta_f\rho) + \frac{1}{N-n}df(\nabla\rho)^2. \tag{12.4}$$

For $a, b \in R$ and $\lambda > 0$, the inequality $\left(\sqrt{\frac{\lambda}{\lambda+1}}\, a + \sqrt{\frac{\lambda+1}{\lambda}}\, b\right)^2 \geq 0$ implies

$$(a+b)^2 \geq \frac{1}{\lambda+1}a^2 - \frac{1}{\lambda}b^2.$$

By taking $a = \Delta_f\rho$, $b = df(\nabla\rho)$ and $\lambda = (N-n)/(n-1)$, we get

$$(\Delta\rho)^2 \geq \frac{n-1}{N-1}(\Delta_f\rho)^2 - \frac{n-1}{N-n}df(\nabla\rho)^2.$$

Then it follows from (12.4) that

$$0 \geq \frac{(\Delta_f\rho)^2}{N-1} + \mathrm{Ric}_f^N(\nabla\rho) + \nabla\rho(\Delta_f\rho). \tag{12.5}$$

This is similar to (12.3).

Let $m(x)$ denote the mean curvature of the metric sphere $\rho^{-1}(t) = S(p, t)$. Let $c_y(t) = \exp_p(ty)$ be a minimizing geodesic over $[0, i_y]$. Let $m(t) := m(c_y(t))$. We have

$$m(t) = \Delta\rho|_{c_y(t)}.$$

Then

$$m'(t) = \nabla\rho(\Delta\rho)|_{c_y(t)}.$$

It follows from (12.3) that

$$m'(t) + \frac{m(t)^2}{n-1} + \mathrm{Ric}(c_y'(t)) \leq 0. \tag{12.6}$$

Assume that $\mathrm{Ric}(c_y'(t))) \geq (n-1)H$. Then

$$m'(t) + \frac{m(t)^2}{n-1} + (n-1)H \leq 0.$$

Let

$$m_H(t) := \frac{d}{dt} \ln \Big[s_H(t)^{n-1} \Big].$$

It satisfies

$$m_H'(t) + \frac{m_H(t)^2}{n-1} + (n-1)H = 0.$$

Note that

$$m(t) = \frac{n-1}{t} + o(t), \qquad m_H(t) = \frac{n-1}{t} + o(t).$$

Lemma 12.1. *Assume that for $0 < t < i_y$,*

$$\mathrm{Ric}(c_y'(t))) \geq (n-1)H.$$

Then

$$m(t) \leq m_H(t).$$

We now estimate the weighted mean curvature:

$$m_f(t) := \Delta_f \rho|_{c_y(t)} = m(t) - df(c_y'(t)).$$

For a constant H and δ, let

$$m_{H,\delta}(t) := \frac{d}{dt} \ln \Big[e^{\delta t} [s_H(t)]^{n-1} \Big] = m_H(t) + \delta.$$

Under the assumption (12.7),

$$m_f(t) \leq m_{H,\delta}(t).$$

We get the following

Proposition 12.2. *Assume that*

$$\mathrm{Ric}(x, \nabla\rho)) \geq (n-1)H, \qquad df(c_y'(t)) \leq -\delta. \tag{12.7}$$

Then

$$m_f(t) \leq m_{H,\delta}(t).$$

It follows from (12.6) that

$$m_f'(t) + \frac{m(t)^2}{n-1} + \mathrm{Ric}_f^\infty(c_y'(t)) \leq 0. \tag{12.8}$$

Assume that

$$\mathrm{Ric}_f^\infty(c_y'(t)) \geq (n-1)H.$$

We obtain a rough estimate from (12.8) that

$$m_f'(t) \leq -(n-1)H.$$

Then for $r > \rho_o$,

$$m_f(r) \leq m_f(\rho_o) - (n-1)H(r - \rho_o).$$

We obtain the following

Proposition 12.3. ([10]) *Let (M, g, dV) be a complete Riemannian manifold with $dV = e^{-f}dV_g$. Assume that*

$$\mathrm{Ric}_f^{\infty}(\nabla\rho) \geq (n-1)H.$$

Then

$$\Delta_f \rho \leq m_o - (n-1)H(\rho(x) - \rho_o),$$

where $m_o := \sup_{\rho(x)=\rho_o} \Delta_f \rho(x)$.

It is proved in [10] that under the assumption

$$\mathrm{Ric}_f^{\infty}(c_y'(t)) \geq (n-1)H, \qquad df(c_y'(t)) \geq -\delta,$$

the mean curvature $m_f(t)$ is bounded above by

$$m_f(t) \leq m_{H,\delta}(t).$$

When $H > 0$, the above estimate holds only for $t \leq \frac{\pi}{2\sqrt{H}}$.

Proposition 12.4. ([10]) *Let (M, g, dV) be a complete Riemannian manifold with $dV = e^{-f}dV_g$. Suppose that for any $x \in M$,*

$$\mathrm{Ric}_f^{\infty}(x, \nabla\rho_x) \geq (n-1)H, \qquad df_x(\nabla\rho_x) \geq -\delta.$$

Then the distance function $\rho(x) = d(p, x)$ on Ω_p satisfies

$$\Delta_f \rho \leq m_{H,\delta}(\rho).$$

When $H > 0$, the above estimate holds on the set where $\rho(x) \leq \pi/(2\sqrt{H})$.

It follows from (12.5) that

$$m_f'(t) + \frac{m_f(t)^2}{N-1} + \mathrm{Ric}_f^N(c_y'(t)) \leq 0.$$

Assume that

$$\mathrm{Ric}_f^N(\nabla\rho) \geq (N-1)H.$$

Then

$$m_f'(t) + \frac{m_f(t)^2}{N-1} + (N-1)H \leq 0.$$

Let

$$m_H^N(t) := \frac{d}{dt}\Big[\ln s_H(t)^{N-1}\Big].$$

It satisfies

$$(m_f^N)'(t) + \frac{(m_f^H(t))^2}{N-1} + (N-1)H = 0.$$

Note that

$$m_f(t) = \frac{n-1}{t} + o(t), \qquad m_H^N(t) = \frac{N-1}{t} + o(t).$$

Then

$$\lim_{t\to 0^+} \Big\{ m_f(t) - m_H^N(t) \Big\} \leq 0.$$

Let

$$h(t) := \Big\{ m_f(t) - m_H^N(t) \Big\} e^{-\int_t^{\varepsilon} \frac{m(\tau)+m_H^N(\tau)}{N-1} d\tau}.$$

Then $h'(t) \leq 0$. This implies that $h(t) \leq 0$ for $t > 0$. Then we obtain the following

$$m_f(t) \leq m_H^N(t).$$

Proposition 12.5. ([8]) *Let (M, g, dV) be a complete Riemannian manifold with $dV = e^{-f}dV_g$. Assume that*

$$\mathrm{Ric}_f^N(\nabla\rho) \geq (N-1)H.$$

Then the distance function $\rho(x) = d(p, x)$ satisfies

$$\Delta_f \rho \leq m_H^N(\rho).$$

Corollary 12.6. *Let (M, g, dV) be a complete Riemannian manifold with $dV = e^{-f} dV_g$. Assume that*

$$\mathrm{Ric}_f^{\infty}(\nabla\rho) \geq (n-1)H, \quad |df(\nabla\rho)| \leq \delta.$$

then for any $N > n$, the distance function $\rho(x) = d(p, x)$ satisfies

$$\Delta_f \rho \leq m_K^N(\rho),$$

where $K := \frac{1}{N-1}\{(n-1)H + \frac{1}{N-n}\delta^2\}$.

13. Comparison Theorems in Finsler Geometry

We are now going to give some applications to the Laplacian of a distance function on a positively complete Finsler metric measure manifold (M, F, dV). Let $p \in M$ and $\rho(x) = d(p, x)$ be the distance function. Then $\nabla\rho$ is a geodesic field on $\Omega_p = M \setminus \{Cut(p)\}$. Let $\hat{g} := g_{\nabla\rho}$ be the induced Riemannian metric. Then $dV = e^{-f} dV_{\hat{g}}$, where $f(x) = \tau(x, \nabla\rho_x)$ the distortion of (F, dV) in the direction of $\nabla\rho_x$. The function $\rho(x) = d(p, x)$ is also a distance function of $\hat{g}$ with $\nabla\rho = \hat{\nabla}\rho$. Moreover, we have the following relationship between the geometric quantities of (F, dV) and that of $(\hat{g}, dV)$.

$$S(\nabla\rho) = df(\hat{\nabla}\rho), \qquad \dot{S}(\nabla\rho) = \mathrm{Hess}(f)(\hat{\nabla}\rho) \tag{13.1}$$

$$\mathrm{Ric}(\nabla\rho) = \widehat{\mathrm{Ric}}(\hat{\nabla}\rho), \tag{13.2}$$

$$\Delta\rho = \hat{\Delta}_f\rho,$$

It follows from (13.1) and (13.2) that

$$\begin{aligned} \mathrm{Ric}^N(\nabla\rho) &= \mathrm{Ric}(\nabla\rho) + \dot{S}(\nabla\rho) - \frac{1}{N-n} S(\nabla\rho) \\ &= \widehat{\mathrm{Ric}}(\hat{\nabla}\rho) + \widehat{\mathrm{Hess}}(f)(\hat{\nabla}\rho) - \frac{1}{N-n} df(\hat{\nabla}\rho) \\ &= \widehat{\mathrm{Ric}}_f^N(\hat{\nabla}\rho). \end{aligned}$$

Similarly, we have

$$\mathrm{Ric}^{\infty}(\nabla\rho) = \widehat{\mathrm{Ric}}_f^{\infty}(\hat{\nabla}\rho).$$

In virtue of Propositions 12.2, 12.3, 12.4 and 12.5, we obtain the following three theorems for Finsler metric measure manifolds.

Theorem 13.1. *Let (M, F, dV) be a positively complete Finsler metric measure manifold. Suppose that*

$$\mathrm{Ric}(x, \nabla\rho_x) \geq (n-1)H, \qquad S(x, \nabla\rho) \geq -\delta.$$

Then the distance function $\rho(x) = d(p, x)$ on Ω_p satisfies

$$\Delta\rho \leq m_{H,\delta}(\rho).$$

Theorem 12.2 is proved by the author ([9]). It is reduced to Proposition 12.4 when F is Riemannian.

By Proposition 12.3, one can easily obtain the following theorem. This theorem can be used to show that positively complete Finsler manifolds with $\mathrm{Ric}^{\infty} \geq (n-1)H > 0$ must have finite volume [2].

Theorem 13.2. *Let (M, F, dV) be a positively complete Finsler metric measure manifold. Suppose that for some $N > n$,*

$$\mathrm{Ric}^{\infty}(x, \nabla\rho_x) \geq (n-1)H.$$

Then the distance function $\rho(x) = d(p, x)$ on $\Omega_p \setminus B(p, \rho_o)$ satisfies

$$\Delta\rho(x) \leq m_o - (n-1)H(\rho(x) - \rho_o).$$

where $m_o := \sup_{x \in \rho^{-1}(\rho_o)} \Delta\rho(x)$.

Now we state two important theorems. One can easily show them by Propositions 12.4 and 12.5 above.

Theorem 13.3. ([12]) *Let (M, F, dV) be a positively complete Finsler metric measure manifold. Suppose that for any point $x \in M$,*

$$\mathrm{Ric}^{\infty}(x, \nabla\rho_x) \geq (n-1)H, \qquad S(x, \nabla\rho_x) \geq -\delta,$$

Then the distance function $\rho(x) = d(p, x)$ on Ω_p satisfies

$$\Delta\rho \leq m_{H,\delta}(\rho). \tag{13.3}$$

When $H > 0$, the above estimate holds on $\Omega_p \cap B(p, \pi/(2\sqrt{H}))$.

Theorem 13.4. ([6]) *Let (M, F, dV) be a positively complete Finsler metric measure manifold. Suppose that for some $N > n$,*

$$\mathrm{Ric}^{N}(x, \nabla\rho_x) \geq (N-1)H$$

then the distance function $\rho(x) = d(p, x)$ on Ω_p satisfies

$$\Delta\rho \leq m^N_H(\rho). \tag{13.4}$$

Using the estimates in (13.3) and (13.4), one can obtain volume comparison theorems of Bishop-Gromov type.

14. Estimates on Injectivity Values

Assume that for some $N > n$, the Ricci curvature satisfies

$$\mathrm{Ric}^{N}(x, \nabla\rho_x) \geq (N-1)H.$$

Then by Theorem 13.4, the following holds on Ω_p,

$$\Delta\rho(x) \leq m^N_H(\rho(x)) := \frac{d}{dt}[\ln\chi(t)]|_{t=\rho(x)},$$

where

$$\chi(t) = [s_H(t)]^{N-1}, \qquad 0 < t < t_o.$$

where $t_o := +\infty$ if $H \leq 0$ and $t_o := \pi/\sqrt{H}$ if $H > 0$. By Lemma 9.1, we obtain the following

Theorem 14.1. *Let (M, F, dV) be a positively complete Finsler metric measure manifold. Assume that for some $N > n$,*

$$\mathrm{Ric}^N \geq (N-1)H > 0.$$

Then

$$i_y \leq \frac{\pi}{\sqrt{H}} \qquad \forall y \in S_pM.$$

In particular, $\mathrm{Diam}(M) \leq \pi/\sqrt{H}$.

By Theorem 14.1, one can easily prove the following

Theorem 14.2. *([2]) Let (M, F, dV) be a positively complete Finsler metric measure manifold. Assume that*

$$\mathrm{Ric}^{\infty} \geq K > 0, \qquad |S| \leq \delta.$$

then

$$i_y \leq \frac{\pi}{\sqrt{K}}\Big\{\frac{\delta}{\sqrt{K}} + \sqrt{\frac{\delta^2}{K} + n - 1}\Big\}.$$

Proof: Let $N > N_o := n + \frac{\delta^2}{K}$. Under the assumption,

$$\mathrm{Ric}^N \geq (N-1)H,$$

where

$$H := \frac{K(N-n) - \delta^2}{(N-1)(N-n)}.$$

Then

$$\mathrm{Diam}(M) \leq \frac{\pi}{\sqrt{H}}.$$

Viewing H as a function of N, we see that

$$\sup_{N > N_o} H = \frac{K}{\Big(\frac{\delta}{\sqrt{K}} + \sqrt{\frac{\delta^2}{K} + n - 1}\Big)^2}.$$

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