Weighted Ricci curvature in Riemann-Finsler geometry

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ABSTRACT: Ricci curvature is one of the important geometric quantities in Riemann-Finsler geometry. Together with the S-curvature, one can define a weighted Ricci curvature for a pair of Finsler metric and a volume form on a manifold. One can build up a bridge from Riemannian geometry to Finsler geometry via geodesic fields. Then one can estimate the Laplacian of a distance function and the mean curvature of a metric sphere under a lower weighted Ricci curvature by applying the results in the Riemannian setting. These estimates also give rise to a volume comparison of Bishop-Gromov type for Finsler metric measure manifolds.

1. Introduction

On a complete Riemannian manifold \((M, g)\) with a volume form \(dV = e^{-f}dV_g\), we have the so-called weighted Laplacian \(\Delta_f u\) and weighted Ricci curvature \(\text{Ric}_f^N\) defined by

\[
\Delta_f u = \text{div}(\nabla u) = \Delta u - df(\nabla u),
\]

\[
\text{Ric}_f^N = \text{Ric}_g + \text{Hess}_g(f) - \frac{1}{N-n}(df)^2.
\]

Set \(\text{Ric}_f^\infty := \text{Ric}_g + \text{Hess}_g(f)\). In literatures, \(\text{Ric}_f^N\) is called the \(N\)-Bakry-Emery Ricci tensor and \(\text{Ric}_f^\infty\) the Bakry-Emery Ricci tensor. In 1997, Z. Qian gave an upper bound on the weighted Laplacian \(\Delta_f \rho\) of a distance function \(\rho(x) = d(p, x)\) under a lower weighted Ricci curvature bound: \(\text{Ric}_f^N \geq (N - 1)H\). Using the upper bound \(\Delta_f \rho\), he generalized the Bishop-Gromov volume comparison to weighted volume ([8]). Later, G. Wei and W. Wylie gave an estimate on \(\Delta_f \rho\) under other Ricci curvature bounds \(\text{Ric}_f^\infty \geq (n - 1)H\) and \(df \geq -\delta\). These results can be applied to Finsler metric measure manifolds after we build up a bridge from the Riemannian setting to the non-Riemannian setting.

Finsler metrics are just Riemannian metrics without quadratic restriction. The notions of Riemann curvature and Ricci curvature in Riemann geometry are naturally extended to Finsler geometry. Every Finsler metric \(F\) on a manifold \(M\) induces a spray \(G\) which is a special vector field on the tangent bundle \(TM\). The geodesics of \(F\) are characterized as the projections of the integral curves of \(G\). The Riemann curvature and some other

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non-Riemannian quantities such as the Berwald curvature and the \( \chi \)-curvature are defined by the spray \( G \). There are quantities such as the Cartan torsion and the Landsberg curvature are defined by the Finsler metric \( F \) and its spray \( G \).

On a Finsler metric measure manifold \( (M,F,dV) \), the non-linear weighted Laplacian \( \Delta = \Delta_{(F,dV)} \) is defined in a natural way:

\[
\Delta u = \operatorname{div}(\nabla u),
\]
where \( \nabla u \) denotes the gradient of \( u \) with respect to \( F \) and \( \operatorname{div}(\cdot) \) denotes the divergence of a vector field with respect to the volume form \( dV \).

The weighted Ricci curvature \( \text{Ric}^N = \text{Ric}^N_{(F,dV)} \) of \( (F,dV) \) are defined by the Ricci curvature \( \text{Ric} \) of \( F \) and the S-curvature \( S \) of \( (F,dV) \).

\[
\text{Ric}^N = \text{Ric} + \dot{S} - \frac{1}{N-n} S^2.
\]
This weighted Ricci curvature is first studied by S. Ohta [6].

My motivation to write this survey article is to build up a bridge from Riemannian geometry to non-Riemannian geometry via geodesic fields. Then many comparison theorems on Riemannian manifolds with a volume form can be carried over to Finsler manifolds with a volume form. This goal can be achieved due to the fact that the Riemann curvature \( R_Y \) of a Finsler metric \( F \) can be expressed as the Riemann curvature \( R_Y = R(\cdot,Y)Y \) of the induced Riemannian metric \( \hat{g} := g_Y \) by a geodesic field \( Y \). Thus their Ricci curvatures are equal in the direction of \( Y \), \( \text{Ric}(Y) = \text{Ric}(Y) \), as the trace of their Riemann curvature \( R_Y \) and \( R_Y \), respectively. For a volume form \( dV \), one has the notion of distortion \( \tau = \tau(x,y) \). The S-curvature is the rate of change of \( \tau \) along a geodesic. In the direction of \( Y_x, dV = e^{-f(x)}dV_{\hat{g}} \), where \( f(x) = \tau(x,Y_x) \). Then \( \Delta = \Delta_f \) where \( \text{Hess}(f) \) denotes the Hessian of \( f \) with respect to \( \hat{g} \). Further, \( S(x,Y_x) = Y_x[\tau(\cdot,Y)] = df(Y_x) \) and \( \dot{S}(x,Y_x) = Y_x[S(\cdot,Y)] = \dot{\text{Hess}}(f)(Y_x) \). Therefore

\[
\text{Ric}^N(x,\nabla \rho_x) = \text{Ric}_f(x,\nabla \rho_x),
\]
where \( \text{Ric}^N_f \) denotes the weighted Ricci curvature of \( (\hat{g},dV = e^{-f}dV_{\hat{g}}) \). Therefore estimates on \( \Delta_f \rho \) under a lower bound \( \text{Ric}^N_f \geq (N-1)H \) will be carried over to \( \Delta \rho \) under a lower bound \( \text{Ric}^N \geq (N-1)H \). That is, the results in [8][10] give rise to estimates on the Laplacian \( \Delta \rho \) under certain Ricci curvature bounds ([7]).

### 2. Finsler Metrics

A Finsler metric \( F \) on a manifold \( M \) is a \( C^\infty \) function on \( TM \setminus \{0\} \) with the following properties:

(a) \( F(x,\lambda y) = \lambda F(x,y) \), \( \lambda > 0 \).
(b) \( g_{ij}(x,y) := \frac{1}{2}[F^2]_{ij}(x,y), y \neq 0 \), is positive definite.

By (a) and (b), one can get

\[
F(x,y_1+y_2) \leq F(x,y_1) + F(x,y_2), \quad y_1,y_2 \in T_xM.
\]

Thus at every point \( x \in M, F_x := F|_{T_xM} \), is a norm on \( T_xM \). The norm \( F_x \) induces a family of inner products \( g_y \) on \( T_xM \):

\[
g_y(u,v) = g_{ij}(x,y)u^iv^j, \quad u = u^i \frac{\partial}{\partial x^i}|_x, \quad v = v^i \frac{\partial}{\partial x^i}|_x.
\]

The length of a curve \( c: [a,b] \to M \) is given by

\[
L(c) := \int_a^b F(c(t),c'(t))dt.
\]

Locally minimizing curves with constant speed are characterized by

\[
\frac{d^2x^i}{dt^2} + 2G^i \left( x, \frac{dx}{dt} \right) = 0,
\]
where

\[
G^i(x,y) = \frac{1}{4} g^{il}(x,y) \left( \frac{\partial g_{jk}}{\partial x^l}(x,y) + \frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{kl}}{\partial x^j}(x,y) \right) y^j y^k.
\]

The local functions \( G^i = G^i(x,y) \) form a global vector field \( G \) on \( TM \):

\[
G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.
\]
**G** is called the *spray* of **F**.

Put
\[
N^i_j := \frac{\partial G^i}{\partial y^j}, \quad \Gamma^i_{jk} := \frac{\partial^2 G^i}{\partial y^j \partial y^k}.
\]

We modify the natural local frame \( \{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \} \) by
\[
\delta \delta x^i := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}.
\]

The local dual frame \( \{ dx^i, \delta y^i \} \) is given by \( \delta y^i := dy^i + N^i_j dx^j \).

The tangent space of \( TM \) at \( y \in T_x M \setminus \{0\} \) has a natural decomposition
\[
T_y(TM) = H_y(TM) \oplus V_y(TM),
\]
where
\[
H_y(TM) := \text{span} \{ \delta \delta x^i \}, \quad V_y(TM) := \text{span} \{ \frac{\partial}{\partial y^i} \}.
\]

For \( X = X^i \frac{\partial}{\partial x^i} \in C^\infty(TM) \) and \( y \in T_x M \), define
\[
D_y X = \left\{ dX^i(y) + X^j N^i_j(x,y) \right\} \frac{\partial}{\partial x^i}|_x.
\]

\( D \) is usually called a *non-linear connection*. If \( F = \sqrt{g_{ij}(x) y^i y^j} \) is Riemannian, \( D \) becomes a linear connection on \( TM \). It is the well-known *Levi-Civita connection*.

Let \( \omega^i := dx^i \), \( \omega^{n+i} := \delta y^i \) and \( \omega_j^i := \Gamma_{jk}^i(x,y) dx^k \).

Then we get the first set of structure equations
\[
d\omega^i = \omega^j \wedge \omega_j^i.
\]

The local curvature forms are defined by
\[
\Omega_j^i := d\omega_j^i - \omega_{jk}^i \wedge \omega_k^j.
\]

We can express \( \Omega_j^i \) as
\[
\Omega_j^i := \frac{1}{2} R_j^{i kl} \omega^k \wedge \omega^l - B_j^{i kl} \omega^k \wedge \omega^{n+l}
\]
with \( R_j^{i kl} + R_j^{i lk} = 0 \). We obtain two important curvatures: the Riemann curvature tensor \( R_j^{i kl} \) and the Berwald curvature tensor \( B_j^{i kl} \). Put
\[
R_k^i := y^j R_j^{i kl} y^l.
\]

Then
\[
R_j^{i kl} = \frac{1}{3} \left\{ R_j^{i kl} - R_j^{i kl} \right\}.
\]

We obtain a family of linear maps \( R_y : T_x M \rightarrow T_x M \),
\[
R_y(u) = R^i_k(x,y) a^k \frac{\partial}{\partial x^i}|_x, \quad u = a^k \frac{\partial}{\partial x^k}|_x \in T_x M.
\]

It is called the *Riemann curvature*. In local coordinates, \( R^i_k \) can be expressed by
\[
R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]
3. Geodesic Fields

We shall describe the Riemann curvature via geodesic fields from the Riemann-geometry point view. Let \((M, F)\) be a Finsler manifold. A vector field \(Y\) on an open subset \(U \subset M\) is called a geodesic field if every integral curve \(c(t)\) of \(Y\) in \(U\) is a geodesic of \(F\):

\[
c'(t) = Y_{c(t)}.
\]

In local coordinates, a geodesic field \(Y = Y^i \frac{\partial}{\partial x^i}\) are characterized by

\[
Y^j(x) \frac{\partial Y^i}{\partial x^j}(x) + 2G^i(x, Y_x) = 0. \tag{3.1}
\]

Here we identify \(Y_x = Y^i(x) \frac{\partial}{\partial x^i}|_x\) with \((Y^1(x), \cdots, Y^n(x))\). For any non-zero vector \(y \in T_x M\), there is an open neighborhood \(U_x\) and a geodesic field \(Y\) on \(U_x\) such that \(Y_x = y\). \(Y\) is called a geodesic extension of \(y\). The geodesic vector field \(Y\) on \(U\) induces a Riemannian metric \(\hat{g} := g_Y\) on \(U\).

\[
\hat{g}_z(u, v) := g_Y(u, v), \quad z \in U, u, v \in T_z U.
\]

Let \(\hat{D}\) denote the Levi-Civita connection of \(\hat{g}\) on \(U\). We have the following

**Lemma 3.1.** In local coordinates \((x^i, y^i)\) in \(TM\),

\[
\hat{G}^j(x, Y_x) = G^j(x, Y_x), \quad \hat{N}^j(x, Y_x) = N^j(x, Y_x), \tag{3.2}
\]

**Proof:** \(N^j := \frac{\partial \hat{G}^j}{\partial y^j}\) are given by

\[
N^j = \frac{1}{2} \hat{g}^{ij} \left\{ \frac{\partial \hat{g}_{jk}}{\partial x^i} + \frac{\partial \hat{g}_{ij}}{\partial x^k} - \frac{\partial \hat{g}_{jk}}{\partial x^i} \right\} y^k - 2\hat{g}^{ij} C_{jkl} G^k. \tag{3.3}
\]

Note that

\[
\frac{\partial \hat{g}_{ij}}{\partial x^k}(x) = \frac{\partial \hat{g}_{ij}}{\partial x^k}(x, Y_x) + 2C_{ijl}(x, Y_x) \frac{\partial Y^l}{\partial x^k}(x). \tag{3.4}
\]

It follows from (3.3) and (3.4) that

\[
\hat{N}^j(x, Y_x) = \frac{1}{2} \hat{g}^{ij}(x) \left\{ \frac{\partial \hat{g}_{jk}}{\partial x^i}(x, Y_x) + \frac{\partial \hat{g}_{ij}}{\partial x^k}(x, Y_x) - \frac{\partial \hat{g}_{jk}}{\partial x^i}(x) \right\} Y^k(x)
\]

\[
= \frac{1}{2} \hat{g}^{ij}(x, Y_x) \left\{ \frac{\partial \hat{g}_{jk}}{\partial x^i}(x, Y_x) + \frac{\partial g_{ij}}{\partial x^k}(x, Y_x) - \frac{\partial g_{jk}}{\partial x^i}(x, Y_x) \right\} Y^k(x) - 2\hat{g}^{ij}(x, Y_x) C_{jkl}(x, Y_x) G^k(x, Y_x)
\]

\[
= N^j(x, Y_x).
\]

This gives (3.2). By (3.2), we obtain

\[
2G^j(x, Y_x) = N^j(x, Y_x)Y^j(x) = \hat{N}^j(x, Y_x)Y^j(x) = 2\hat{G}^j(x, Y_x).
\]

Thus \(Y\) also satisfies

\[
Y^j(x) \frac{\partial Y^i}{\partial x^j}(x) + 2\hat{G}^i(x, Y_x) = 0.
\]

Thus \(Y\) is also a geodesic field of \(\hat{F}\). \(\square\)

**Lemma 3.2.** Let \(Y\) be a geodesic field of \(F\) on an open subset \(U\) and \(\hat{g} := g_Y\). Then \(Y\) is also a geodesic field of \(\hat{g}\).

**Proof:** \(Y\) satisfies (3.1). By Lemma 3.1, \(Y\) satisfies

\[
Y^j(x) \frac{\partial Y^i}{\partial x^j}(x) + 2\hat{G}^i(x, Y_x) = 0.
\]

Thus \(Y\) is a geodesic field of \(\hat{g}\) too. \(\square\)
Lemma 3.3. Let $Y$ be a geodesic field of $F$ on an open subset $U$ and $\tilde{g} := g_Y$. For any vector field $W$ on $U$,  

$$D_Y W = \tilde{D}_Y W.$$ 

Proof: 

$$D_Y W = \left\{ Y_z(W^i) + W^j(x)N^i_j(x, Y_z) \right\} \frac{\partial}{\partial x^i}|_{c(t)}$$ 

$$= \left\{ Y_z(W^i) + W^j(x)N^i_j(x, Y_z) \right\} \frac{\partial}{\partial x^i}|_{c(t)} = \tilde{D}_Y W.$$ 

One can use the non-linear connection $D$ to define the covariant derivative $D_c X(t)$ of a vector field $X = X^i(t) \frac{\partial}{\partial x^i}|_{c(t)}$ along a curve $c(t)$, $a \leq t \leq b$. 

$$D_c X(t) := \left\{ \frac{dX^i}{dt}(t) + X^j(t)N^i_j(x(t), x'(t)) \right\} \frac{\partial}{\partial x^i}|_{c(t)}.$$ 

Let $c(t), a \leq t \leq b$ be a geodesic in $(M, F)$. 

$$J(t)$$ satisfies the Jacobi equation: 

$$D_c D_c J(t) + R_c(J(t)) = 0, $$ 

where $R_c$ is the Riemann curvature in the direction of $\dot{c}$. 

We may assume that $H$ is an embedding and $\frac{dH}{ds}(s, t)$ can be extended to a geodesic field $Y$ in a neighborhood $U$ of $c$ so that 

$$Y_{H(s, t)} = \frac{dH}{ds}(s, t).$$ 

Note that $Y_{c(t)} = c'(t)$. Since each $c_s(t) = H(s, t)$ is a geodesic of $\tilde{g}$ by Lemma 3.2, $J(t)$ is a Jacobi field of $\tilde{g}$. Let $\tilde{R}$ denote the Riemann curvature of $\tilde{g} := g_Y$. It is proved in Riemann geometry that $J(t)$ satisfies 

$$D_c \tilde{D}_c J(t) + \tilde{R}_c(J(t)) = 0.$$ 

By Lemma 3.3, $D_c = \tilde{D}_c$, we get 

$$R_c(J(t)) = \tilde{R}_c(J(t)).$$ 

We conclude that $R_y = \tilde{R}_y$. We obtain the following 

Lemma 3.4. Let $y \in T_x M \setminus \{0\}$ and $Y$ be a geodesic extension of $y$. Let $\tilde{R}$ denote the Riemann curvature of $\tilde{g} := g_Y$. Then $R_y = \tilde{R}_y$. Moreover, 

$$g_y(R_y(u, v)) = g_y(u, R_y(v)), \quad u, v \in T_x M.$$ 

Proof: It has been proved in Riemannian geometry that 

$$\tilde{g}_x(\tilde{R}_y(u, v)) = \tilde{g}_x(u, \tilde{R}_y(v)), \quad u, v \in T_x M.$$ 

Note that $\tilde{g}_x = g_y$ and $\tilde{R}_y = R_y$. This completes the proof. 

The Ricci curvature $\text{Ric}(x, y)$ is the trace of the Riemann curvature $R_y : T_x M \to T_x M$. 

$$\text{Ric}(x, y) := \text{trace}(R_y) = \sum_{i=1}^n R^i_i(x, y).$$ 

We have the following 

Lemma 3.5. Let $y \in T_x M \setminus \{0\}$ and $Y$ be a geodesic extension of $y$. Let $\tilde{\text{Ric}}$ denote the Ricci curvature of $\tilde{g} := g_Y$. Then 

$$\text{Ric}(x, y) = \tilde{\text{Ric}}(x, y).$$
4. Volume Form and S-curvature

Let \((M, F)\) be a Finsler manifold. The Finsler metric \(F\) determines a distance function \(d_F\). The distance function \(d_F\) determines the Hausdorff measure \(\mu_F\). H. Busemann finds a volume form \(dV_F\) for the Hausdorff measure

\[
\mu_F(U) = \int_U dV_F,
\]

In local coordinates \((x^i)\), \(dV_F = \sigma_F(x)dx^1 \cdots dx^n\), is given by

\[
\sigma_F(x) = \frac{\text{Vol}(B^n(1))}{\text{Vol}\{(y^i) \in \mathbb{R}^n \mid F(x, y) < 1\}}.
\]

**Example 4.1.** Let \(F(x, y) = \sqrt{g_{ij}(x)y^iy^j}\) be a Riemannian metric. The Busemann-Hausdorff volume form \(dV_F = \sigma_F(x)dx^1 \cdots dx^n\) is given by

\[
\sigma_F(x) = \sqrt{\det(g_{ij}(x))}.
\]

**Example 4.2.** Let \(F = \alpha(x, y) + \beta(x, y)\) be a Randers metric on \(M\) where

\[
\alpha(x, y) = \sqrt{a_{ij}(x)y^iy^j}, \quad \beta(x, y) = b_i(x)y^i,
\]

with

\[
b(x) := \sqrt{a_{ij}(x)b_i(x)b_j(x)} < 1,
\]

where \((a^{ij}(x)) = (a_{ij}(x))^{-1}\). The Busemann-Hausdorff volume \(dV_F = \sigma_F(x)dx^1 \cdots dx^n\) is given by

\[
\sigma_F(x) = \left(1 - b(x)^2\right)^{\frac{n+1}{2}} \sigma_\alpha(x),
\]

where \(\sigma_\alpha(x) = \sqrt{\det(a_{ij}(x))}\).

A Randers metric \(F = \alpha + \beta\) can be also expressed in the following navigation form:

\[
F = \frac{\sqrt{(1 - \beta^2)\hat{a}^2 + \beta^2}}{1 - \beta^2} \left(1 - b(x)^2\right)^{\frac{n+1}{2}} \sigma_\alpha(x),
\]

where \(\hat{a} = \sqrt{a_{ij}(x)y^iy^j}\) is a Riemannian metric and \(\hat{\beta} = b_i(x)y^i\) is a 1-form with \(\hat{b}(x) := \|\hat{\beta}_x\|\hat{a} < 1\). We have

\[
dV_F = dV_\hat{a}.
\]

Randers metrics in the form (4.1) are called **general \((\alpha, \beta)\)-metrics**. However, for other general \((\alpha, \beta)\)-metrics, it is impossible to find an explicit formula for the Busemann-Hausdorff volume form.

Let \(dV\) be a volume form on \((M, F)\). In local coordinates

\[
g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^iy^j}(xy), \quad dV = \sigma(x)dx^1 \cdots dx^n.
\]

Then the following quantity is well-defined

\[
\tau(x, y) := \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.
\]

\(\tau = \tau(x, y)\) is called the **distortion** of \((F, dV)\). The vertical covariant derivative is the **mean Cartan torsion**:

\[
I_i(x, y) = \tau'_y(x, y) = g^{jk}(x, y)C_{ijk}(x, y).
\]

Brickell’s Theorem says that for a regular Finsler metric \(F\), \(I = 0\) if and only if \(F\) is Riemannian ([1]).

The derivative of the distortion along a geodesic is the so-called S-curvature

\[
S(x, y) := \left.\frac{d}{dt}\left[\tau(c(t), c'(t))\right]\right|_{t=0}
\]
Thus the S-curvature is also defined for a spray \( G \) and a volume form \( dV \).

Let \( Y \) be a geodesic field on an open subset \( U \) and \( \hat{g} := g_Y \). Observe that

\[
\tau(x, Y_z) = \ln \frac{\sqrt{\det(g_{ij}(x, Y_z))}}{\sigma(x)} = \ln \frac{\sqrt{\det(g_{ij}(x))}}{\sigma(x)}.
\]

Let

\[
f(x) := \tau(x, Y_z), \quad x \in U.
\]

Then

\[
dV = e^{-f(x)} dV_{\hat{g}}.
\]

The S-curvature of \((F, dV)\) is given by

\[
S(x, Y_z) = Y_x[\tau(\cdot, Y)] = df_x(Y_z).
\]

5. The Gradient and Laplacian

For a Finsler metric \( F \) on a manifold \( M \), the dual co-Finsler metric \( F^* \) is a function on \( T^* M \), defined by

\[
F^*(x, \eta) := \sup_{y \in T_x M} \frac{\eta(y)}{F(x, y)}.
\]

Conversely, \( F \) can be viewed as the dual metric to \( F^* \) by the following identity:

\[
F(x, y) = \sup_{\eta \in T^*_y M} \eta(y) = \frac{\eta(y)}{F^*(x, \eta)}.
\]

The Lagrange \( L_x : T_x M \to T^*_x M \) is defined by

\[
L_x(y) := g_y(y, \cdot).
\]

The Lagrange map \( L_x \) is positively homogeneous in \( y \in T_x M \), that is, \( L_x(\lambda y) = \lambda L_x(y) \), \( \forall \lambda > 0 \). Further, \( L_x : T_x M \setminus \{0\} \to T^*_x M \setminus \{0\} \) is a diffeomorphism with

\[
F^*(x, L_x(y)) = F(x, y).
\]

**Definition 5.1.** Let \( f \) be a \( C^\infty \) function on \( M \) and \( x \in M \). If \( df_x \neq 0 \), set

\[
\nabla f_x := L_x^{-1}(df_x).
\]

If \( df_x = 0 \), set

\[
\nabla f_x = 0.
\]

\( \nabla f_x \) is called the gradient of \( f \) at \( x \).

From the definition of \( \nabla f_x \), we have

\[
df_x = L_x(\nabla f_x) = g_{\nabla f_x}(\nabla f_x, \cdot).
\]

\( \nabla f \) is \( C^\infty \) on the open set \( \{df_x \neq 0\} \). Let \( g^{\ast ij}(x, \eta) := \frac{1}{2} \frac{\partial^2}{\partial \eta^i \partial \eta^j} [F^{*2}](x, \eta) \). Then

\[
\nabla f_x = \nabla^i f(x) \frac{\partial}{\partial \xi^i} |_x = g^{\ast ij}(x, df) \frac{\partial f}{\partial \xi^j}(x) \frac{\partial}{\partial \xi^i} |_x.
\]

Let \( f \) be a \( C^\infty \) function on an open subset \( U \subset M \) and \( N_t := f^{-1}(t) \subset U \). Suppose that \( df_x \neq 0 \) at some point \( x \in N_t \), then \( N_t \) is a hypersurface in a neighborhood of \( x \). We have

\[
g_{\nabla f_x}(\nabla f_x, v) = df_x(v) = 0, \quad \forall v \in T_x N_t.
\]

Namely, \( \nabla f_x \) is perpendicular to \( N_t \) at \( x \) with respect to \( g_{\nabla f_x} \).

Assume that \( df_x \neq 0 \) on an open set \( U \). Let \( \hat{g} := g_{\nabla f} \) and \( \hat{\nabla} f \) denote the gradient of \( f \) with respect to \( \hat{g} \). We have the following

\[
\text{where } c(t) \text{ is the geodesic with } c(0) = x \text{ and } c'(0) = y. \text{ In local coordinates, if } G^i = G^i(x, y) \text{ denote the spray coefficients of } F \text{ and } dV = \sigma(x)dx^1 \cdots dx^n, \text{ then}
\]

\[
S(x, y) = \frac{\partial G^m}{\partial y^m}(x, y) - y^m \frac{\partial}{\partial \theta} \ln \sigma(x).
\]

Thus the S-curvature is also defined for a spray \( G \) and a volume form \( dV \).
Lemma 5.2. \( \nabla f = \hat{\nabla} f \). \hfill (5.1)

Further

\[ F(x, \nabla f_x) = \sqrt{g_x(\nabla f_x, \nabla f_x)} = \|\hat{\nabla} f_x\|_g. \]

Proof: For any tangent vector \( v \in T_xM \),

\[ \hat{g}(\hat{\nabla} f, v) = df_x(v) = \hat{g}(\nabla f, v). \]

This implies (5.1).

Observe that

\[ F(x, \nabla f_x)^2 = \hat{g}(\hat{\nabla} f_x, \nabla f_x) = \hat{g}_x(\hat{\nabla} f_x, \hat{\nabla} f_x). \]

□

For a vector \( y \in T_xM \), define the Hessian \( \text{Hess}(f) \) of \( f \) at \( x \) by

\[ \text{Hess}(f)(y) := \frac{d^2}{dt^2} \left[ f \circ c_y(t) \right]_{t=0}, \]

where \( c_y(t) \) is the geodesic with \( c_y(0) = x \) and \( c'_y(0) = y \). \( \text{Hess}(f)(\lambda y) = \lambda^2 \text{Hess}(f)(y), \lambda > 0 \). But \( \text{Hess}(f)(y) \) is not quadratic in \( y \in T_xM \). In local coordinates

\[ \text{Hess}(f)(y) = \nabla_i \nabla_j f(x) y^i y^j - 2G^i(y, x) \frac{\partial f}{\partial x^i}(x). \]

It is easy to prove the following

Lemma 5.3. Let \( f \) be a function on \( M \),

\[ \text{Hess}(f)(y) = y[Y(f)]. \]

where \( Y \) is a geodesic extension of \( y \).

6. Distance Functions

Let \( A \) be a closed subset in a Finsler manifold \( (M, F, dV) \). Let\( \rho_+(x) := d(A, x), \rho_-(x) := -d(x, A). \)

\( \rho_+ \) and \( \rho_- \) are locally Lipschitz functions. Thus they are differentiable almost everywhere. We have the following
Lemma 6.1. Let $\rho(x) = \rho_+(x)$ or $\rho_-(x)$. Assume that $f$ is $C^\infty$ on an open subset $U \subset M$. Then

$$F^*(x, dp_x) = F(x, \nabla \rho_x) = 1, \quad x \in U.$$ 

Therefore we make the following

Definition 6.2. A Lipschitz function $f$ on a Finsler manifold $(M, F)$ is called a distance function if the following identity holds almost everywhere on $M$

$$F(x, \nabla f_x) = 1.$$ 

Let $f$ be a $C^\infty$ function on an open subset $U \subset M$ with $df_x \neq 0$, $\forall x \in U$. Let $\hat{g} := g_{\nabla f}$ be the induced Riemannian metric on $U$. By Lemma 5.2, we have

$$\nabla f = \hat{\nabla} f.$$ 

Further, $F(x, \nabla f_x) = \sqrt{\hat{g}(x, \nabla f_x)}$. Thus $F(x, \nabla f_x) = 1$ if and only if $\hat{g}(x, \nabla f_x) = 1$. That is, $f$ is a distance function of $F$ if and only if $\hat{g}$ is a distance function of $f$.

Proposition 6.3. Let $\rho = \rho(x)$ be a $C^\infty$ distance function on $U \subset M$. Let $\hat{g} := g_{\nabla \rho}$. Then $\nabla \rho = \hat{\nabla} \rho$ is a geodesic field of $F$ and $\hat{g}$.

Proof: $\rho$ is a distance function of $F$. We have

$$g_{\nabla \rho}(\nabla \rho, \nabla \rho) = 1.$$ 

Let $V$ be a vector field on $U$ such that $V \perp \nabla \rho$ with respect to $g_{\nabla \rho}$ and $[V, \nabla \rho] = 0$. One can choose $V$ in the following way. Take a variation $H : (-\varepsilon, \varepsilon) \times (a, b) \to M$ such that

$$\nabla \rho|_{H(s,t)} = \frac{\partial H}{\partial t}(s,t), \quad V_{H(s,t)} = \frac{\partial H}{\partial s}(s,t).$$ 

We can make $V \perp \nabla \rho$ with respect to $g_{\nabla \rho}$. Then

$$0 = V g_{\nabla \rho}(\nabla \rho, \nabla \rho) = 2 g_{\nabla \rho}(\nabla \rho, D_{\nabla \rho} V) = -2 g_{\nabla \rho}(D_{\nabla \rho} \nabla \rho, V).$$

Thus $D_{\nabla \rho} \nabla \rho = 0$. This implies that $\nabla \rho$ is a geodesic field of $F$. By a similar argument, one can show that $\hat{\nabla} \rho$ is a geodesic field of $\hat{g}$. \hfill \Box

7. Mean Curvature

Let $i : N \to M$ be an embedded hypersurface in a Finsler manifold $(M, F)$ and $dV$ be a volume form on $M$. Let $n$ be a normal vector to $N$ at $x \in N$,

$$g_{n}(n,v) = 0, \quad \forall v \in T_{x,N}.$$ 

Let $e_{1} = n, e_{2}, \ldots, e_{n}, 2 \leq a \leq n$ be an orthonormal basis for $(T_{x,M}, g_{n})$. Let $\theta^{1}, \ldots, \theta^{n}$ be the dual basis for $T^{*}M$. Then $\theta^{i}(v) = 0$ for all $v \in T_{x,N}$ and $\{i^{*} \theta^{a}\}_{a=1}^{n}$ is a basis for $T_{x}N$.

Let $dV_{x} = \sigma(x) \theta^{1} \wedge \cdots \wedge \theta^{n}$ at $x \in N \subset M$. The induced volume form $dA_{x}$ at $x \in N$ is given by

$$dA_{x} = \sigma(x) i^{*} \theta^{2} \wedge \cdots \wedge i^{*} \theta^{n}.$$ 

Locally, $N$ can be viewed as a level surface of a distance function $\rho$ so that $n = \nabla \rho|_{N}$ is a normal vector to $N$. We may assume that $N$ is contained in an open subset $U \subset M$ and it divides $U$ into two disconnected open subsets $U_{-}$ and $U_{+}$. Thus $U = U_{-} \cup N \cup U_{+}$. Define $\rho : U \to \mathbb{R}$ by

$$\rho(x) := d(N,x), \quad x \in U_{+}, \quad \rho(x) := -d(x,N), \quad x \in U_{-}.$$ 

$\rho$ is a $C^\infty$ distance function on $U$ with $N = \rho^{-1}(0)$. Then $\rho$ has the required property.

For $x \in N$, let $c_{\varepsilon}(t)$ denote the integral curve of $\nabla \rho$ with $c_{\varepsilon}(0) = x$. For a small $\varepsilon > 0$, let $N_{\varepsilon} := \rho^{-1}(\varepsilon)$ and define $\phi_{\varepsilon} : N \to N_{\varepsilon}$ by

$$\phi_{\varepsilon}(x) = c_{\varepsilon}(\varepsilon). \quad (7.1)$$ 

Let $dA_{x}$ denote the induced volume form on $N_{\varepsilon}$. Let $dA = dA_{0}$. Set

$$\phi_{\varepsilon}^{*} dA_{x} = \Theta_{\varepsilon}(x) dA.$$ 

Note that $\Theta_{0}(x) = 1$. 

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Definition 7.1. The mean curvature of $N$ at $x \in N$ is defined by

$$m(x) := \frac{d}{d\varepsilon} \left[ \ln \Theta_{\varepsilon}(x) \right]_{\varepsilon=0}.$$ 

In the following, we are going to show that the Laplacian of a distance function is the mean curvature of the level surface of the distance function.

Let $\rho$ be a $C^\infty$ distance function on an open subset $U \subset M$. Let $N := N_0 := \rho^{-1}(0)$. There is a local coordinate system $(t, x^a)$ such that $(x^a)$ is a local system on $N$ and

$$\rho(t, x^a) = t.$$ 

We have

$$\nabla \rho = \frac{\partial}{\partial t}.$$ 

Let $x^1 := t$ and $\nabla \rho = \nabla^i \rho \frac{\partial}{\partial x^i}$. Then

$$\nabla^1 \rho = 1, \quad \nabla^a \rho = 0 \quad (a = 2, \cdots, n).$$ 

Let

$$\theta^1 = dt, \quad \theta^a = dx^a.$$ 

Put $dV = \sigma(t, x^a) dt \wedge dx^2 \wedge \cdots \wedge dx^n$. Then the induced volume form $dA$ on $N_t$ is given by

$$dA|_{N_t} = \sigma(t, x^a) dx^2 \wedge \cdots \wedge dx^n.$$ 

The Laplacian $\Delta \rho$ of $\rho$ can be expressed as

$$\Delta \rho = \frac{\partial}{\partial x^1} (\nabla^1 \rho) + \nabla^1 \rho \frac{\partial}{\partial x^1} (\ln \sigma) = \nabla^1 \rho \frac{\partial}{\partial x^1} (\ln \sigma).$$ 

We obtain

$$\Delta \rho(x) = \frac{\partial}{\partial t} \left[ \ln \sigma(t, x^a) \right]_{t=\rho(x)}.$$ 

The map $\phi_\varepsilon : N_t \to N_{t+\varepsilon}$ is given by

$$\phi_\varepsilon(t, x^a) = (t + \varepsilon, x^a).$$ 

The pull-back volume form $\phi_\varepsilon^* dA|_{N_{t+\varepsilon}} = \Theta_{\varepsilon}(t, x^a) dA_N$ is given by

$$\phi_\varepsilon^* dA|_{N_{t+\varepsilon}} = \sigma(t + \varepsilon, x^a) dx^2 \wedge \cdots \wedge dx^n = \frac{\sigma(t + \varepsilon, x^a)}{\sigma(t, x^a)} dA_N.$$ 

That is

$$\Theta_{\varepsilon}(t, x^a) = \frac{\sigma(t + \varepsilon, x^a)}{\sigma(t, x^a)}.$$ 

Therefore the mean curvature $m(x)$ at $x \in N_t$ is given by

$$m(x) = \frac{d}{d\varepsilon} \left[ \frac{\sigma(t + \varepsilon, x^a)}{\sigma(t, x^a)} \right]_{\varepsilon=0} = \frac{\partial}{\partial t} \left[ \ln \sigma(t, x^a) \right].$$ 

We have proven the following

Lemma 7.2. The Laplacian of a distance function $\rho$ on $U$ is the mean curvature of the level surface.

$$\Delta \rho|_x = m(x), \quad x \in N := \rho^{-1}(0).$$
8. Volume of Geodesic Balls

In this section, we are going to express the volume of metric balls an integral over the unit tangent sphere at a point. Let \((M,F)\) be a positively complete manifold. Let \(p \in M\) and \(\exp_p : T_pM \to M\) be the exponential map so that \(\exp_p(T_pM) = M\). The exponential map \(\exp_p\) is \(C^\infty\) on \(T_pM \setminus \{0\}\) and \(C^1\) at \(0 \in T_pM\) such that

\[
(\exp_p)_*|_0 : T_0(T_pM) \equiv T_pM \to T_pM
\]

is an identity map.

For a unit vector \(y \in S_pM\), the conjugate value \(c_y\) of \(y\) is the first zero \(r\) of a Jacobi field \(J(t)\) along \(c(t) = \exp_p(ty)\), equivalently, the smallest positive number \(r > 0\) such that \(d(\exp_p)|_{ty} : T_y(T_pM) \to T_{\exp_p(ty)}M\) is singular.

For a vector \(y \in S_pM\), the injectivity value \(i_y\) of \(y\) is the largest possible value \(r\) such that \(c_y|[0,c]\) is a minimizing geodesic. The cut-domain \(\Omega_p\) is defined by

\[
\Omega_p := \{ \exp_p(ty) \mid y \in S_pM, 0 \leq t < i_y \}.
\]

\(\Omega_p\) is a star-shaped open domain in \(M\). Further the cut-locus \(\text{Cut}(p) := M \setminus \Omega_p\) has zero measure. Let \(T\Omega_p :=\{ty \in T_pM, \ 0 < t < i_y, y \in S_pM\}\). \(\exp_p\) is a diffeomorphism from \(T\Omega_p\) to an open subset \(\Omega_p = \exp_p(T\Omega_p)\). Let \(dV_p\) denote the restriction of \(dV\) on \(T_pM\). It induces a volume form \(dA_p\) on the unit tangent sphere \(S_pM\). Define a map \(\phi : [0,\infty) \times S_pM \to M\) by

\[
\phi(t,y) = \exp_p(ty).
\]

Let \(S^*_pM := \{ y \in S_pM \mid r < i_y \}\) and \(\tilde{S}(p,r) := S(p,r) \cap \Omega_p\). Then \(\varphi_t = \phi(r,\cdot) : S^*_pM \to \tilde{S}(p,r)\) is a diffeomorphism. Let

\[
\varphi^*_tdA_{\tilde{S}(p,r)} = \eta_r(p,y)dA_p, \quad y \in S^*_pM.
\]

Recall the map \(\phi_x : S(p,r) \to S(p,r+\varepsilon)\) defined in (7.1) and put

\[
\phi^*_xdA_{S(p,r+\varepsilon)} = \Theta_x(y)dA_{\tilde{S}(p,r)}.
\]

It is easy to see that and \(x = \exp_p(ry)\),

\[
\Theta_x(x) = \frac{\eta_{r+\varepsilon}(p,y)}{\eta_r(p,y)}.
\]

Then the mean curvature at \(x\) is given by

\[
m(x) = \frac{d}{dr} \ln \Theta_x(x)|_{r=0} = \frac{d}{dr} \ln \eta_r(p,y).
\]

The volume of \(\tilde{S}(p,r)\) and \(B(p,R) \setminus B(p,r)\) can be expressed as an integral of \(\eta_r(p,y)\).

\[
\text{Vol}(\tilde{S}(p,r)) = \int_{\tilde{S}(p,r)} dA_{\tilde{S}(p,r)} = \int_{S^*_pM} \varphi^*_tdA_{\tilde{S}(p,r)} = \int_{S^*_pM} \eta_r(p,y)dA_p.
\]

\[
\text{Vol}(B(p,R) \setminus B(p,r)) = \int_r^R \text{Vol}(\tilde{S}(p,t))dt = \int_r^R \int_{S^*_pM} \eta_t(p,y)dA_pdt.
\]

Therefore, estimates on the mean curvature \(m(x)\) along a geodesic \(c_y(t) = \exp_p(ty)\) will gives estimates on \(\text{Vol}(\tilde{S}(p,r))\) and then on \(\text{Vol}(B(p,R) \setminus B(p,r))\).

9. Curvature-free comparison theorems

Let \(0 \leq \rho_o < t_o \leq +\infty\) and a \(C^\infty\) function

\[
\chi : (\rho_o, t_o) \to (0, +\infty), \tag{9.1}
\]

if \(\rho_o = 0\) then \(\lim_{\iota \to 0^+} \chi(t) = 0\) and if \(t_o \leq +\infty\), then \(\lim_{\iota \to t_o^-} \chi(t) = 0\).

A typical example is tat \(\chi(t) = e^{\delta s_{H}(t)}\), \(0 < \rho_o < t < t_o\), where \(s_{H}(t) = f(t)\) is the unique solution to the following equation:

\[
f''(t) + Hf(t) = 0, \quad f(0) = 0, \quad f'(0) = 1.
\]

\(t_o := \pi/\sqrt{H}\) if \(H > 0\) and \(t_o = +\infty\) if \(H \leq 0\).
Lemma 9.1. Let \( \chi \) be a function in (9.1). Assume that for \( x \in [B(p, t_o) \setminus B(p, \rho_o)] \cap \Omega_p \)

\[
\Delta \rho_x \leq \frac{d}{dt} \left[ \ln \chi(t) \right]_{t=\rho(x)}.
\]

Then the injectivity value \( i_y \leq i_o \) for any \( y \in S_p M \).

Proof: By assumption we have

\[
\frac{d}{dt} \ln \eta_t(p, y) \leq \frac{d}{dt} \ln \chi(t), \quad 0 < t < \min(t_o, i_y).
\]

For a sufficiently small \( \varepsilon > 0 \),

\[
\eta_t(p, y) \leq \chi(t) \frac{\eta_{\rho_o + \varepsilon}(p, y)}{\chi(\rho_o + \varepsilon)}, \quad 0 < \varepsilon < t < \min(t_o, i_y).
\]

We claim that \( i_y \leq t_o \). Suppose it is not true, i.e., \( t_o < i_y \). Then

\[
0 < \eta_t(p, y) \leq \chi(t) \frac{\eta_{\rho_o + \varepsilon}(p, y)}{\chi(\rho_o + \varepsilon)}, \quad 0 < \eta < i_o < i_y.
\]

Letting \( t \to t_o^- \), one gets

\[
\eta_{t_o}(p, y) \leq 0.
\]

This is impossible. Therefore \( i_y \leq i_o \).

Therefore we make the following

**Assumption X**: Let \( \chi \) be a function defined in (9.1). For a point \( p \in M \), \( d_p := \sup_{x \in M} d(p, x) \leq t_o \) and

\[
\Delta \rho(x) = m(x) \leq \frac{d}{dt} \left[ \ln \chi(t) \right]_{t=\rho(x)}, \quad x \in \Omega_p \setminus B(p, \rho_o).
\]

Under **Assumption X**, one can easily see that along any minimizing geodesic \( c_y(t) = \exp_p(ty) \), \( \rho_o < t < i_y \),

\[
\frac{d}{dt} \left[ \ln \eta_t(p, y) \right] \leq 0.
\]

Thus for any \( \rho_o < \tau < t \),

\[
\eta_t(p, y)\chi(\tau) \leq \eta_\tau(p, y)\chi(t).
\]

Integrating the above inequality over \( S_p^1 M \), we get

\[
\text{Vol}(\tilde{S}(p, t))\chi(\tau) \leq \text{Vol}(\tilde{S}(p, \tau))\chi(t).
\]

Integrating the above identity with respect to \( \tau \) over \([\rho_o, \rho] \), we get

\[
\text{Vol}(\tilde{S}(p, t)) \int_{\rho_o}^\rho \chi(\tau) d\tau \leq \text{Vol}(B(p, \rho) \setminus B(p, \rho_o))\chi(t)
\]

Integrating the above inequality with respect to \( t \) over \([\rho, r] \), we get

\[
\text{Vol}(B(p, r) \setminus B(p, \rho_o)) \int_{\rho_o}^\rho \chi(\tau) d\tau \leq \text{Vol}(B(p, r) \setminus B(p, \rho_o)) \int_{\rho}^r \chi(t) dt
\]

Adding \( \text{Vol}(B(p, \rho) \setminus B(p, \rho_o)) \int_{\rho_o}^\rho \chi(\tau) d\tau \) to both sides yields

\[
\text{Vol}(B(p, r) \setminus B(p, \rho_o)) \int_{\rho_o}^\rho \chi(\tau) d\tau \leq \text{Vol}(B(p, t) \setminus B(p, \rho_o)) \int_{\rho}^r \chi(t) dt
\]

We have the following
Proposition 9.2. ([2]) Let \( \chi : (\rho_0, t_o) \to (0, +\infty) \). Assume that for a point \( p \in M \), Assumption X holds. Then for \( \rho_0 < \rho < r < t_o \),
\[
\frac{\text{Vol}(\mathcal{S}(p, r))}{\text{Vol}(B(p, \rho_0) \setminus B(p, r))} \leq \frac{\chi(r)}{\int_{\rho_0}^{r}\chi(s)ds}.
\]
\[
\frac{\text{Vol}(B(p, r) \setminus B(p, \rho_0))}{\text{Vol}(B(p, \rho) \setminus B(p, \rho_0))} \leq \frac{\int_{\rho_o}^{r}\chi(t)dt}{\int_{\rho_o}^{\rho}\chi(s)ds}.
\]
Thus
\[
\text{Vol}(\mathcal{S}(p, r)) \geq \chi(r)Vol(B(p, \rho_0) \setminus B(p, r)) \geq \int_{\rho_0}^{r}\chi(s)ds.
\]
For a unit vector \( y \in S_p M \), the mean curvature \( m(r) := m|_{c_y(r)} \) of \( S(p, r) \) at \( c_y(r) \) has the following expansion
\[
m(r) = \frac{n - 1}{r} - S(x, y) - \frac{1}{3}[\text{Ric}(y) + 3\hat{S}(x, y)]r + o(r).
\]
This is given in Proposition 14.4.5 in [9]. This gives
\[
\eta_r(p, y) = r^{n-1}\left(1 - S(p, y)r + O(r^2)\right).
\]
Thus
\[
\text{Vol}(S(p, r)) = \phi(p)\text{Vol}(S^{-1}(1))r^{n-1}\left\{1 - s(p)r + O(r^2)\right\}
\]
\[
\text{Vol}(B(p, r)) = \phi(p)\text{Vol}(B^n(1))r^n\left\{1 - \frac{n}{n+1}s(p)r + O(r^2)\right\}.
\]
where \( dV = \phi(x)dV_{BH} \) and \( s(p) := \frac{1}{\text{Vol}(S^n(1))}\int_{S_{p,M}} S(p, y)dA_p \).

10. Weighted Laplacian
Let \( (M, F, dV) \) be a Finsler metric measure manifold. Let \( \rho \) be a \( C^\infty \) distance function on \( U \subset M \). Let \( \hat{g} := g_{\nabla \rho} \). \( \tau \) and \( \hat{\tau} \) denote the distortion of \( (F, dV) \) and \( (\hat{g}, dV) \), respectively. Let \( f(x) = \tau(x, \nabla \rho_x) \) and \( \hat{f}(x) := \hat{\tau}(x, \nabla \rho_x) \). We have
\[
f(x) = \ln \frac{\sqrt{\det(\hat{g}_{ij}(x, \nabla \rho))}}{\sigma(x)} = \ln \frac{\sqrt{\det(g_{ij}(x))}}{\sigma(x)} = \hat{f}(x).
\]
Thus
\[
dV = e^{-f(x)}d\hat{V}_\hat{g}.
\]
Lemma 10.1. The S-curvature and its dot derivative of \( S \) in the direction \( \nabla \rho \) are given by
\[
S(x, \nabla \rho_x) = df(\nabla \rho_x), \quad \hat{S}(x, \nabla \rho_x) = \hat{\text{Hess}}(f)(\nabla \rho_x).
\]
where \( f(x) := \tau(x, \nabla \rho_x) \).

Proof: It follows that
\[
S(x, \nabla \rho_x) = \nabla \rho_x[\tau(\cdot, \nabla \rho)] = \nabla \rho_x(f) = df(\nabla \rho_x).
\]
\[
\hat{S}(x, \nabla \rho_x) = \nabla \rho_x[S(\cdot, \nabla \rho)] = \nabla \rho_x[df(\nabla \rho)] = \hat{\text{Hess}}(f)(\nabla \rho_x).
\]

We now study the Laplacian \( \Delta \rho = \dot{\tau}(\nabla \rho) \) of a distance function \( \rho = \rho(x) \) with respect to \( (F, dV) \). Let \( \hat{\Delta}_f \rho \) denote the Laplacian of \( \rho \) with respect to \( (\hat{g}, dV = e^{-f}dV_{\hat{g}}) \), where \( f(x) := \tau(x, \nabla \rho_x) \).

Lemma 10.2.
\[
\Delta \rho = \hat{\Delta}_f \rho.
\]
Proof: Let \( dV = \sigma(x)dx^1 \cdots dx^n \) and \( dV_{\hat{g}} = \hat{\sigma}(x)dx^1 \cdots dx^n \). Then
\[
f(x) = \tau(x, \nabla \rho_x) = \ln \frac{\hat{\sigma}(x)}{\sigma(x)}.
\]

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\[ \Delta \rho = \frac{\partial}{\partial x^i}(\nabla^i \rho) + \nabla^i \rho \frac{\partial}{\partial x^i} \ln \sigma \]

\[ = \frac{\partial}{\partial x^i}(\tilde{\nabla}^i \rho) + \tilde{\nabla}^i \rho \frac{\partial}{\partial x^i} \ln \tilde{\sigma} \]

\[ = \frac{\partial}{\partial x^i}(\tilde{\nabla}^i \rho) + \tilde{\nabla}^i \rho \left[ \frac{\partial}{\partial x^i} \ln \tilde{\sigma} - \frac{\partial}{\partial x^i} \ln \tilde{\sigma} \right] \]

\[ = \tilde{\Delta} \rho - \tilde{\nabla}^i \rho \frac{\partial}{\partial x^i} \ln \tilde{\sigma} \]

\[ = \tilde{\Delta} \rho - \tilde{\nabla} \rho (f) \]

\[ = \tilde{\Delta} \rho - df(\nabla \rho) = \tilde{\Delta} \rho. \]

\[ \square \]

11. Weighted Ricci Curvature

Let \( F = F(x, y) \) be a Finsler metric and \( dV = \sigma dx^1 \cdots dx^n \) be a volume form on an \( n \)-manifold \( M \). Let \( \text{Ric} = \text{Ric}_F \) denote the Ricci curvature of \( F \) and \( S = S_{(F,dV)} \) denote the S-curvature of \( (F,dV) \). The weighted Ricci curvature is defined by

\[ \text{Ric}^N := \text{Ric} + \dot{S} - \frac{1}{N-n} S^2. \]

\[ \text{Ric}^\infty := \text{Ric} + \text{Hess}(f). \]

If \( F \) is Riemannian, \( dV = e^{-f} dV_F \). Then

\[ S = df. \]

Thus

\[ \text{Ric}^N = \text{Ric} + \text{Hess}(f) - \frac{1}{N-n} (df)^2. \]

\[ \text{Ric}^\infty = \text{Ric} + \text{Hess}(f). \]

This is the well-known weighted Ricci curvature in Riemannian geometry. We are going to show that the weighted Ricci curvature of \((F,dV)\) can be expressed as the weighted Ricci curvature of \((\hat{g},dV)\) in the direction of \( Y \) where \( \hat{g} = g_Y \) is the induced Riemannian metric induced by a geodesic field \( Y \) on an open subset.

Let \( Y \) be a \( C^\infty \) geodesic field on an open subset \( U \subset M \) and \( \hat{g} = g_Y \). Let

\[ dV := e^{-f} dV_{\hat{g}}. \]

where \( f \) is given by

\[ f(x) = \ln \frac{\sqrt{\det(\hat{g}_{ij}(x))}}{\sigma(x)} = \ln \frac{\sqrt{\det(g_{ij}(x,Y_x))}}{\sigma(x)} = \tau(x,Y_x). \]

Here \( \tau = \tau(x,y) \) is the distortion of \( F \) at \( x \).

By the definition of the S-curvature, we have

\[ S(x,y) = y[\tau(\cdot,Y)] = df(y). \]

\[ \dot{S}(x,y) = y[S(\cdot,Y)] = y[Y(f)] = \text{Hess}(f)(y) = \text{Hess}(f)(y). \]

That is, for \( y = Y_x \in T_x M \),

\[ S(x,y) = df(y), \quad \dot{S}(x,y) = \text{Hess}(f)(y). \]

By Lemma 3.5,

\[ \text{Ric}(x,Y_x) = \text{Ric}(x,Y_x). \]

Then in the direction of \( Y_x \)

\[ \text{Ric}(x,Y_x) + \dot{S}(x,Y_x) - \frac{1}{N-n} S(x,Y_x)^2 = \text{Ric}(x,Y_x) + \text{Hess}(f)(Y_x) - \frac{1}{N-n} [df(Y_x)]^2. \]

This proves the following

**Lemma 11.1.** Let \( Y \) be a geodesic field on an open subset \( U \) and \( \hat{g} = g_Y \). Put \( dV = e^{-f} dV_{\hat{g}} \). Then

\[ \text{Ric}^N(x,Y_x) = \text{Ric}^N_f(x,Y_x). \]
12. Comparison Theorems in Riemannian geometry

Let \((M, g, dV = e^{-f}dV_o)\) be a Riemannian metric measure manifold. Let \(\Delta\) denote the Laplacian with respect to \(g\) and \(\Delta_f\) denote the Laplacian with respect to \((g, dV)\), i.e., for a \(C^\infty\) function \(u\) on \(M\),

\[
\Delta_f u = \text{div}_V(\nabla u) = \Delta u - df(\nabla u).
\]

We have the following Bochner formula

\[
\frac{1}{2} \Delta |\nabla u|^2 = |\text{Hess}(u)|^2 + \text{Ric}(\nabla u) + g(\nabla u, \nabla(\Delta u)). \tag{12.1}
\]

Using

\[
\frac{1}{2} df(\nabla(|\nabla u|^2)) = g(\nabla u, \nabla(df(\nabla u))) - |\text{Hess}(f)(\nabla u)|^2,
\]

one gets from (12.1) that

\[
\frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess}(u)|^2 + \text{Ric}_f(\nabla u) + g(\nabla u, \nabla(\Delta_f u)) + \frac{1}{N - n} df(\nabla u)^2. \tag{12.2}
\]

Let

\[
\text{Ric}_f^\infty := \text{Ric} + \text{Hess}(f),
\]

\[
\text{Ric}_f^N := \text{Ric} + \text{Hess}(f) - \frac{1}{N - n} (df)^2
\]

Let \(\rho(x) := d(p, x)\) be the distance function from a point \(p \in M\) so that \(||\nabla \rho|| = 1\). Clearly

\[
\text{Hess}(\nabla \rho) = 0.
\]

Thus

\[
(\Delta \rho)^2 \leq (n - 1)\text{Hess}(\rho).
\]

Letting \(u = \rho(x)\) in (12.1), we obtain

\[
0 \geq \frac{(\Delta \rho)^2}{n - 1} + \text{Ric}(\nabla \rho) + \nabla \rho(\Delta \rho). \tag{12.3}
\]

Letting \(u = \rho(x)\) in (12.2), we obtain

\[
0 \geq \frac{(\Delta \rho)^2}{n - 1} + \text{Ric}_f(\nabla \rho) + \nabla \rho(\Delta_f \rho) + \frac{1}{N - n} df(\nabla \rho)^2. \tag{12.4}
\]

For \(a, b \in R\) and \(\lambda > 0\), the inequality

\[
\left(\sqrt{\frac{a}{\lambda}} + \sqrt{\frac{b}{\lambda}}\right)^2 \geq 0
\]

implies

\[
(a + b)^2 \geq \frac{1}{\lambda + 1} a^2 - \frac{1}{\lambda} b^2.
\]

By taking \(a = \Delta_f \rho, b = df(\nabla \rho)\) and \(\lambda = (N - n)/(n - 1)\), we get

\[
(\Delta \rho)^2 \geq \frac{n - 1}{N - 1} (\Delta_f \rho)^2 - \frac{n - 1}{N - n} df(\nabla \rho)^2.
\]

Then it follows from (12.4) that

\[
0 \geq \frac{(\Delta_f \rho)^2}{N - 1} + \text{Ric}_f(\nabla \rho) + \nabla \rho(\Delta_f \rho). \tag{12.5}
\]

This is similar to (12.3).

Let \(m(x)\) denote the mean curvature of the metric sphere \(\rho^{-1}(t) = S(p, t)\). Let \(c_y(t) = \exp_p(ty)\) be a minimizing geodesic over \([0, i_y]\). Let \(m(t) := m(c_y(t))\). We have

\[
m(t) = \Delta \rho|_{c_y(t)}.
\]

Then

\[
m'(t) = \nabla \rho(\Delta \rho)|_{c_y(t)}.
\]
It follows from (12.3) that
\[ m'(t) + \frac{m(t)^2}{n-1} + \text{Ric}(c_y'(t)) \leq 0. \] (12.6)

Assume that \( \text{Ric}(c_y'(t)) \geq (n-1)H \). Then
\[ m'(t) + \frac{m(t)^2}{n-1} + (n-1)H \leq 0. \]

Let
\[ m_H(t) := \frac{d}{dt} \ln \left( s_H(t)^{n-1} \right). \]
It satisfies
\[ m'_H(t) + \frac{m_H(t)^2}{n-1} + (n-1)H = 0. \]
Note that
\[ m(t) = \frac{n-1}{t} + o(t), \quad m_H(t) = \frac{n-1}{t} + o(t). \]

**Lemma 12.1.** Assume that for \( 0 < t < i_y \),
\[ \text{Ric}(c_y'(t)) \geq (n-1)H. \]

Then
\[ m(t) \leq m_H(t). \]

We now estimate the weighted mean curvature:
\[ m_f(t) := \Delta_f \rho|_{c_y(t)} = m(t) - df(c_y'(t)). \]

For a constant \( H \) and \( \delta \), let
\[ m_{H,\delta}(t) := \frac{d}{dt} \ln \left( e^{\delta t} [s_H(t)]^{n-1} \right) = m_H(t) + \delta. \]
Under the assumption (12.7),
\[ m_f(t) \leq m_{H,\delta}(t). \]

We get the following

**Proposition 12.2.** Assume that
\[ \text{Ric}(x, \nabla \rho) \geq (n-1)H, \quad df(c_y'(t)) \leq -\delta. \] (12.7)

Then
\[ m_f(t) \leq m_{H,\delta}(t). \]

It follows from (12.6) that
\[ m'_f(t) + \frac{m(t)^2}{n-1} + \text{Ric}^\infty(c_y'(t)) \leq 0. \] (12.8)
Assume that
\[ \text{Ric}^\infty(c_y'(t)) \geq (n-1)H. \]

We obtain a rough estimate from (12.8) that
\[ m'_f(t) \leq -(n-1)H. \]
Then for \( r > \rho_o \),
\[ m_f(r) \leq m_f(\rho_o) - (n-1)H(r - \rho_o). \]
We obtain the following
Proposition 12.3. ([10]) Let $(M, g, dV)$ be a complete Riemannian manifold with $dV = e^{-f}dV_g$. Assume that 
\[ \text{Ric}^\infty_f(\nabla \rho) \geq (n-1)H. \]
Then 
\[ \Delta_f \rho \leq m_o - (n-1)H(\rho(x) - \rho_o), \]
where $m_o := \sup_{\rho(x) = \rho_o} \Delta_f \rho(x)$.

It is proved in [10] that under the assumption 
\[ \text{Ric}^\infty_f(c'_y(t)) \geq (n-1)H, \quad df(c'_y(t)) \geq -\delta, \]
the mean curvature $m_f(t)$ is bounded above by 
\[ m_f(t) \leq m_{H, \delta}(t). \]
When $H > 0$, the above estimate holds only for $t \leq \frac{\pi}{2\sqrt{H}}$.

Proposition 12.4. ([10]) Let $(M, g, dV)$ be a complete Riemannian manifold with $dV = e^{-f}dV_g$. Suppose that for any $x \in M$, 
\[ \text{Ric}^\infty_f(x, \nabla \rho_x) \geq (n-1)H, \quad df_x(\nabla \rho_x) \geq -\delta. \]
Then the distance function $\rho(x) = d(p, x)$ on $\Omega_p$ satisfies 
\[ \Delta_f \rho \leq m_{H, \delta}(\rho). \]
When $H > 0$, the above estimate holds on the set where $\rho(x) \leq \pi/(2\sqrt{H})$.

It follows from (12.5) that 
\[ m'_f(t) + \frac{m_f(t)^2}{N-1} + \text{Ric}^\infty_f(c'_y(t)) \leq 0. \]
Assume that 
\[ \text{Ric}^\infty_f(\nabla \rho) \geq (N-1)H. \]
Then 
\[ m'_f(t) + \frac{m_f(t)^2}{N-1} + (N-1)H \leq 0. \]
Let 
\[ m^N_H(t) := \frac{d}{dt} \left[ \ln s_H(t)^{N-1} \right]. \]
It satisfies 
\[ (m^N_H)'(t) + \frac{(m^H(t))'}{N-1} + (N-1)H = 0. \]
Note that 
\[ m_f(t) = \frac{n-1}{t} + o(t), \quad m^N_H(t) = \frac{N-1}{t} + o(t). \]
Then 
\[ \lim_{t \to 0^+} \left\{ m_f(t) - m^N_H(t) \right\} \leq 0. \]
Let 
\[ h(t) := \left\{ m_f(t) - m^N_H(t) \right\} e^{-\int_t^\infty \frac{m(r) + m^N_H(r)}{N-1} dr}. \]
Then $h'(t) \leq 0$. This implies that $h(t) \leq 0$ for $t > 0$. Then we obtain the following 
\[ m_f(t) \leq m^N_H(t). \]

Proposition 12.5. ([8]) Let $(M, g, dV)$ be a complete Riemannian manifold with $dV = e^{-f}dV_g$. Assume that 
\[ \text{Ric}^\infty_f(\nabla \rho) \geq (N-1)H. \]
Then the distance function $\rho(x) = d(p, x)$ satisfies 
\[ \Delta_f \rho \leq m^N_H(\rho). \]
Corollary 12.6. Let \((M, g, dV)\) be a complete Riemannian manifold with \(dV = e^{-f} dV_g\). Assume that
\[
\text{Ric}^\infty_f(\nabla \rho) \geq (n-1)H, \quad |df(\nabla \rho)| \leq \delta.
\]
then for any \(N > n\), the distance function \(\rho(x) = d(p, x)\) satisfies
\[
\Delta_f \rho \leq m^K_N(\rho),
\]
where \(K := \frac{1}{N-1} \{ (n-1)H + \frac{1}{N-n} f^2 \} \).

13. Comparison Theorems in Finsler Geometry

We are now going to give some applications to the Laplacian of a distance function on a positively complete Finsler metric measure manifold \((M, F, dV)\). Let \(p \in M\) and \(\rho(x) = d(p, x)\) be the distance function. Then \(\nabla \rho\) is a geodesic field on \(\Omega_p = M \setminus \{ \text{Cut}(p) \}\). Let \(\hat{g} := g \rho\) be the induced Riemannian metric. Then \(dV = e^{-f} dV_{\hat{g}}\), where \(f(x) = \tau(x, \nabla \rho_x)\) the distortion of \((F, dV)\) in the direction of \(\nabla \rho_x\). The function \(\rho(x) = d(p, x)\) is also a distance function of \(\hat{g}\) with \(\nabla \rho = \nabla \rho\). Moreover, we have the following relationship between the geometric quantities of \((F, dV)\) and that of \((\hat{g}, dV)\).

\[
S(\nabla \rho) = df(\nabla \rho), \quad \hat{S}(\nabla \rho) = \text{Hess}(f)(\nabla \rho)
\]
\[
\text{Ric}(\nabla \rho) = \hat{\text{Ric}}(\nabla \rho),
\]
\[
\Delta \rho = \Delta_f \rho,
\]

It follows from (13.1) and (13.2) that
\[
\text{Ric}^N(\nabla \rho) = \text{Ric}(\nabla \rho) + \hat{S}(\nabla \rho) - \frac{1}{N-n} S(\nabla \rho)
\]
\[
= \hat{\text{Ric}}(\nabla \rho) + \text{Hess}(f)(\nabla \rho) - \frac{1}{N-n} df(\nabla \rho)
\]
\[
= \hat{\text{Ric}}^N_f(\nabla \rho).
\]

Similarly, we have
\[
\text{Ric}^\infty(\nabla \rho) = \hat{\text{Ric}}^\infty_f(\nabla \rho).
\]

In virtue of Propositions 12.2, 12.3, 12.4 and 12.5, we obtain the following three theorems for Finsler metric measure manifolds.

Theorem 13.1. Let \((M, F, dV)\) be a positively complete Finsler metric measure manifold. Suppose that
\[
\text{Ric}(x, \nabla \rho_x) \geq (n-1)H, \quad S(x, \nabla \rho) \geq -\delta.
\]
Then the distance function \(\rho(x) = d(p, x)\) on \(\Omega_p\) satisfies
\[
\Delta \rho \leq m_{H, \delta}(\rho).
\]

Theorem 13.2 is proved by the author ([9]). It is reduced to Proposition 12.4 when \(F\) is Riemannian.

By Proposition 12.3, one can easily obtain the following theorem. This theorem can be used to show that positively complete Finsler manifolds with \(\text{Ric}^\infty \geq (n-1)H > 0\) must have finite volume [2].

Theorem 13.2. Let \((M, F, dV)\) be a positively complete Finsler metric measure manifold. Suppose that for some \(N > n\),
\[
\text{Ric}^\infty(x, \nabla \rho_x) \geq (n-1)H.
\]
Then the distance function \(\rho(x) = d(p, x)\) on \(\Omega_p \setminus B(p, \rho_o)\) satisfies
\[
\Delta \rho(x) \leq m_o - (n-1)H(\rho(x) - \rho_o).
\]
where \(m_o := \sup_{x \in \rho^{-1}(\rho_o)} \Delta \rho(x)\).

Now we state two important theorems. One can easily show them by Propositions 12.4 and 12.5 above.
Theorem 13.3. ([12]) Let \((M, F, dV)\) be a positively complete Finsler metric measure manifold. Suppose that for any point \(x \in M\),
\[
\Ric^\infty(x, \nabla \rho_x) \geq (n - 1)H, \quad S(x, \nabla \rho_x) \geq -\delta,
\]
Then the distance function \(\rho(x) = d(p, x)\) on \(\Omega_p\) satisfies
\[
\Delta \rho \leq m_{H,\delta}(\rho).
\]
(13.3)
When \(H > 0\), the above estimate holds on \(\Omega_p \cap B(p, \pi/(2\sqrt{H}))\).

Theorem 13.4. ([6]) Let \((M, F, dV)\) be a positively complete Finsler metric measure manifold. Suppose that for some \(N > n\),
\[
\Ric^N(x, \nabla \rho_x) \geq (N - 1)H,
\]
then the distance function \(\rho(x) = d(p, x)\) on \(\Omega_p\) satisfies
\[
\Delta \rho \leq m_N^H(\rho).
\]
(13.4)

Using the estimates in (13.3) and (13.4), one can obtain volume comparison theorems of Bishop-Gromov type.

14. Estimates on Injectivity Values

Assume that for some \(N > n\), the Ricci curvature satisfies
\[
\Ric^N(x, \nabla \rho_x) \geq (N - 1)H.
\]
Then by Theorem 13.4, the following holds on \(\Omega_p\),
\[
\Delta \rho(x) \leq m_N^H(\rho(x)) := \frac{d}{dt}[\ln \chi(t)]|_{t=\rho(x)},
\]
where
\[
\chi(t) = [s_H(t)]^{N-1}, \quad 0 < t < t_o,
\]
where \(t_o := +\infty\) if \(H \leq 0\) and \(t_o := \pi/\sqrt{H}\) if \(H > 0\). By Lemma 9.1, we obtain the following

Theorem 14.1. Let \((M, F, dV)\) be a positively complete Finsler metric measure manifold. Assume that for some \(N > n\),
\[
\Ric^N \geq (N - 1)H > 0.
\]
Then
\[
i_y \leq \frac{\pi}{\sqrt{H}} \quad \forall y \in S_p M.
\]
In particular, \(\text{Diam}(M) \leq \pi/\sqrt{H}\).

By Theorem 14.1, one can easily prove the following

Theorem 14.2. ([2]) Let \((M, F, dV)\) be a positively complete Finsler metric measure manifold. Assume that
\[
\Ric^\infty \geq K > 0, \quad |S| \leq \delta.
\]
then
\[
i_y \leq \frac{\pi}{\sqrt{K}} \left\{\frac{\delta}{\sqrt{K}} + \sqrt{\frac{\delta^2}{K} + n - 1}\right\}.
\]
Proof: Let \(N > N_o := n + \frac{\delta^2}{K}\). Under the assumption,
\[
\Ric^N \geq (N - 1)H,
\]
where
\[
H := \frac{K(N - n) - \delta^2}{(N - 1)(N - n)}.
\]
Then
\[
\text{Diam}(M) \leq \frac{\pi}{\sqrt{H}}.
\]
Viewing \(H\) as a function of \(N\), we see that
\[
\sup_{N > N_o} H = \frac{K}{\left(\frac{\delta}{\sqrt{K}} + \sqrt{\frac{\delta^2}{K} + n - 1}\right)^2}.
\]
References


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