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A survey on unicorns in Finsler geometry

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ABSTRACT: This survey is an inspiration of my joint paper with Behzad Najafi published in Science in China. I explain some of interesting results about the unicorn problem.

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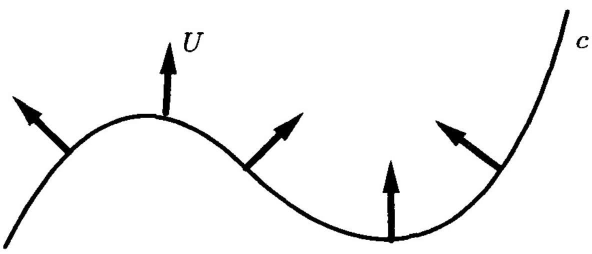
(Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration)

1. Introduction

Let (M, F) be a Finsler manifold. Suppose that $c : [a, b] \to M$ be a piecewise C^∞ curve from $c(a) = p$ to $c(b) = q$.



For every $u \in T_pM$, let us define $P_c : T_pM \to T_qM$ by $P_c(u) := U(b)$, where $U = U(t)$ is the parallel vector field along c such that $U(a) = u$. P_c is called the parallel translation along c.

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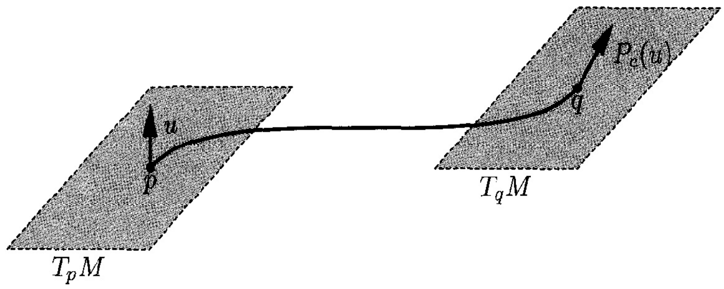
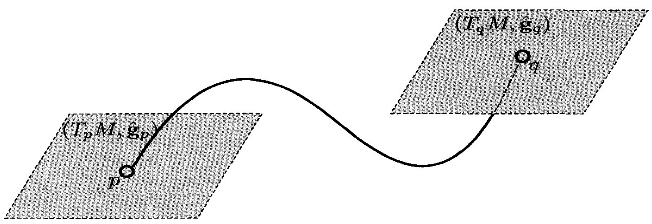


In [15], Ichijyō showed that if F is a Berwald metric then all tangent spaces (T_xM, F_x) are linearly isometric to each other. More precisely, he proved the following.

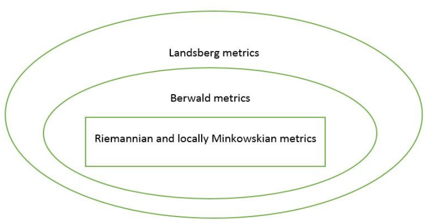
Theorem 1.1. ([15]) *Let (M, F) be a Berwald manifold. For any piecewise smooth curve $c = c(t)$ from p to q in M, the parallel translation $P_c : (T_pM, F_p) \to (T_qM, F_q)$ is a linear isometry.*

Let us consider the Riemannian metric $\hat{g}_x$ on $T_xM_0 := T_xM - \{0\}$ which is defined by $\hat{g}_x := g_{ij}(x, y)\delta y^i \otimes \delta y^j$, where $g_{ij} := 1/2[F^2]_{y^iy^j}$ is the fundamental tensor of F and $\{\delta y^i := dy^i + N^i_j dx^j\}$ is the natural coframe on T_xM associated with the natural basis $\{\partial/\partial x^i|_x\}$ for T_xM. In [16], Ichijyō proved the following.

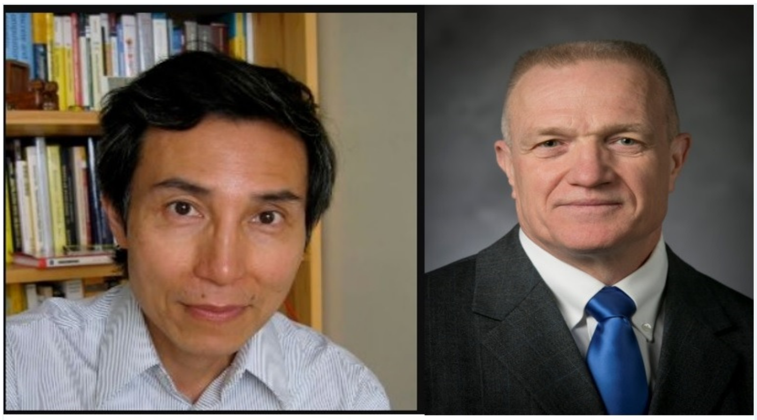
Theorem 1.2. ([16]) *Let (M, F) be a Landsberg manifold. Then for any piecewise smooth curve $c = c(t)$ from p to q in M, the parallel translation P_c along c preserves the induced Riemannian metrics on the slit tangent spaces, i.e., $P_c : (T_PM_0, \hat{g}_p) \to (T_qM_0, \hat{g}_q)$ is an isometry.*



By definition, every Berwald metric is a Landsberg metric, but the converse may not hold.



In 1996, Matsumoto found a list of rigidity results which almost suggest that such a pure Landsberg metric (non-Berwaldian metric) does not exist [19]. In 2003, Matsumoto emphasized this problem again and looked on it as the most important open problem in Finsler geometry. It is a long-existing open problem in Finsler geometry to find non-Berwaldian Landsberg metrics. Bao called such metrics unicorns in Finsler geometry, mythical single-horned horse-like creatures which exist in legend but have never been seen by human beings [3]. There are a lot of unsuccessful attempts to find explicit examples of unicorns. In [7], Bryant has announced that in the two dimensions, there is an abundance of such metrics depending on two families of functions of two variables.



David Bao Robert Bryant

Let (Ω, F) be a spherically symmetric Finsler surface in $\mathbb{R}^2$, where Ω is a domain in $\mathbb{R}^2$. Suppose that $(x_1, x_2) \in \Omega$ and $(x_1, x_2; y_1, y_2) \in T\Omega_0$. Put

$$r = \sqrt{x_1^2 + x_2^2}, \qquad u = \sqrt{y_1^2 + y_2^2}, \qquad s = \frac{x_1 y_1 + x_2 y_2}{u}.$$

Define the Finsler metric F on Ω by following

$$F = u \exp\left(\int_0^s \frac{(c+1)s^2 - \left(2r^2 c_0 - 1\right) s\sqrt{r^2 - s^2} - 2r^2 c}{(r^2 - s^2)\left((2c_0 r^2 - 1)\sqrt{r^2 - s^2} - (c+1)s\right)} ds \right) a(r),$$

where c is a constant, c_0 is a smooth function of r and

$$a(r) = \exp\left(\int -\frac{2c_0 r^2 - 1 + 2c^2 - 2c}{r(2c_0 r^2 - 1)} dr \right)$$

The geodesic spray of F are computed as follows

$$G^i = uPy^i + u^2 Q x^i, \quad i = 1, 2$$

where

$$P = f_1(r)s + f_2(r)\sqrt{r^2 - s^2}, \qquad Q = c_0(r) + c_2(r)s^2 + c_1(r)s\sqrt{r^2 - s^2},$$

Here f_1, f_2, c_1, c_2 are smooth functions of r. In [42], Zhou claimed that (Ω, F) is a singular Landsberg Finsler surface with a vanishing flag curvature which is not Berwaldian. More precisely, he tried to find the required Finsler surfaces among the spherically symmetric metrics defined on a domain in $\mathbb{R}^2$. Unfortunately, Elgendi-Youssef showed that the examples of non-Berwaldian Landsberg surfaces with vanishing flag curvature, obtained by Zhou, are in fact Berwaldian [13]. Consequently, Bryant's claim is still unverified.

As the first step, Asanov found a special family of unicorns in the class of non-regular (α, β)-metrics [2]. By using the Asanov's original notations in [1], the general form of unicorns (the Finsleroid-Finsler metrics in his terminology) on a manifold M is given by

$$F = e^{\frac{G\Phi}{2}} \sqrt{\beta^2 + g\mathcal{R}\beta + \mathcal{R}^2},$$

where $\beta = b_i(x)y^i$ is the Finsleroid axis one form, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric with $||\beta_x||_\alpha := \sqrt{a^{ij}(x)b_i b_j} = 1$ and

$$\mathcal{R} := \sqrt{\gamma_{ij} y^i y^j}, \quad \gamma_{ij} := a_{ij} - b_i b_j,$$

Also, $g := g(p)$ is a scalar function on M with $-2 < g(p) < 2$ and

$$\Phi := \begin{cases} \arctan \frac{G}{2} - \arctan \frac{2\mathcal{R} + g\beta}{2h\beta} + \frac{\pi}{2} & \text{if } \beta > 0, \\ \arctan \frac{G}{2} - \arctan \frac{2\mathcal{R} + g\beta}{2h\beta} - \frac{\pi}{2} & \text{if } \beta < 0. \end{cases}$$

Here,

$$h := \sqrt{1 - \frac{g^2}{4}}, \quad G := \frac{g}{h}.$$

For some positive constant q, Asanov's unicorns can be express as follows

$$\phi(s) = \exp\left[\int_0^s \frac{q\sqrt{b^2 - t^2}}{1 + qt\sqrt{b^2 - t^2}} dt\right]. \tag{1.1}$$

In [23], Sabau-Shibuya-Shimada studied the problem of existence of generalized Landsberg structures on surfaces using the Cartan-Kähler theorem and a path geometry approach.

Hideo Shimada and Sorin Sabau

In [38], Vincze studied those unicorns constructed by Asanov (called Finsleroid-Finsler spaces), showing that they belong to a class of special Finsler spaces, called generalized Berwald spaces, if and only if the Finsleroid charge is constant. In particular, a Finsleroid-Finsler space is a Landsberg space if and only if it is a generalized Berwald manifold with a semi-symmetric compatible linear connection.

Csaba Vincze

In [27], Szabó made an argument to prove that any regular Landsberg metric must be of Berwald type. But unfortunately, there is a little gap in Szabó's argument [27]. As pointed out in Szabó's correction to [27], his argument only applies to the so-called dual Landsberg spaces. Taking into account of so many unsuccessful efforts of so many researchers, one can say that unicorn problem is becoming more and more puzzling.

In order to find a unicorn, one can consider the class of (α, β)-metrics which form a rich class of important and computable Finsler metrics. An (α, β)-metric on a manifold M is defined by $F := \alpha\phi(s)$, $s := \beta/\alpha$, where $\phi = \phi(s)$ is a scalar function on a open set $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Many of (α, β)-metrics with special and important curvature properties have been found and discussed. In [24], Shen proved that dose not exist any unicorn in the class of regular Landsbergian (α, β)-metrics.

Theorem 1.3. (Shen [24]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian regular (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is a Landsberg metric if and only if β is parallel with respect to α. In this case, F is a Berwald metric.*

Zhongmin Shen

For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, let us define $b_{i;j}$ by $b_{i;j}\theta^j := db_i - b_j{\theta^j}_i$, where $\theta^i := dx^i$ and ${\theta^j}_i := {\Gamma^j}_{ik}dx^k$ denote the Levi-Civita connection form of α. Let us define

$$r_{ij} := \frac{1}{2}\left(b_{i;j} + b_{j;i}\right), \quad s_{ij} := \frac{1}{2}\left(b_{i;j} - b_{j;i}\right).$$

In [24], Shen found more complicated family of unicorns in the class of non-regular (α, β)-metrics which contains the Asanov's metrics (1.1).

Theorem 1.4. (Shen [24]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian almost regular (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that $b(x) := ||\beta_x||_\alpha \neq 0$. Then F is a Landsberg metric if and only if either β is parallel with respect to α, in this case, F is a Berwald metric, or ϕ is given by*

$$\phi(s) = c_3 \exp\left[\int_0^s \frac{c_1\sqrt{1-(t/b_0)^2} + c_2 t}{1 + t(c_1\sqrt{1-(t/b_0)^2} + c_2 t)}\, dt\right], \tag{1.2}$$

and β satisfies the following equations:

$$s_{ij} = 0, \qquad r_{ij} = k(b^2 a_{ij} - b_i b_j),$$

where c_1, c_2, c_3 are constants with $c_1 \neq 0$, $1 + c_2 b_0 > 0$ and $c_3 > 0$, and $k = k(x)$ is a scalar function on M. Moreover, F is not a Berwald metric if and only if $k \neq 0$.

The metric F defined by (1.2) is singular in two directions $y = (\pm 1, 0, \cdots, 0) \in T_x\mathbb{R}^n$ at any point x. These examples do not settle the problem because all known examples are y-local.

In [10], Crampin studied the unicorn problem (the Landsberg-Berwald problem in his terminology) in order to find that whether or not there are y-global unicorns. Using the technique of averaging the fundamental tensor over the indicatrix in the case of y-global Landsberg space (the technique introduced by Vincze in [37] for the case of y-global Berwald space), he proved that the averaged Berwald connection is the Levi-Civita connection of the averaged metric.

In [17], Li-Shen gave a complete characterization of 2-dimensional Landsberg (α, β)-metrics.

Theorem 1.5. (Li-Shen [17]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian almost regular (α, β)-metric on a manifold M of dimension $n = 2$. Then F is a Landsberg metric if and only if either β is parallel with respect to α, or β has constant length, $b = b_0$ and ϕ is given by*

$$\phi(s) = \exp\left[\int \frac{P}{1 + sP} ds\right],$$

where

$$P := -\frac{s}{b^2} + \sqrt{1-(s/b)^2}\left\{q_0 + \frac{(1/b^2+q_1)s}{\sqrt{1-(s/b)^2}+(q_0-\lambda/2\sqrt{1/b^2+q_1})s}\right\}.$$

Here, q_0, q_1 and λ are constants. In fact, F is a Berwald metric.

In [30], Tayebi-Najafi classified the class of 3-dimensional (α,β)-metrics with vanishing Landsberg curvature and obtained the following.

Theorem 1.6. (Tayebi-Najafi [30]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian almost regular (α,β)-metric on a manifold M of dimension $n = 3$. Then F is a Landsberg metric if and only if one of the following holds:*

(i) F is a Berwald metric. In this case, F is a Randers metric or a Kropina metric;

(ii) ϕ is given by the ODE

$$\phi^{4-4c}\big(\phi - s\phi'\big)^{4-c}\Big[\phi - s\phi' + (b^2-s^2)\phi''\Big]^{-c} = e^{k_0}, \tag{1.3}$$

where c is a nonzero real constant, k_0 is a real number and $b := ||\beta||_\alpha$. In this case, F is a Berwald metric (regular case) or an almost regular unicorn.

The ODE (1.3) might be solvable in general, but the authors have not been able to prove this yet. If $c = 4/3$, then we get

$$\phi(s) = c_3 \exp\left[\int_0^s \frac{\sqrt{b^2-t^2}+c_1(\kappa-1)t}{c_1 b^2 + t\big[\sqrt{b^2-t^2}+c_1(\kappa-1)t\big]}\, dt\right],$$

where c_1 and c_2 are real constants, $c_3 > 0$ and $\kappa := \sqrt[4]{e^{-3k_0}}$.

In [17], Li-Shen characterized almost regular weakly Landsberg (α,β)-metrics on a manifold M of dimension $n \geq 3$. They have also shown that there exist almost regular weakly Landsberg metrics which are not Landsberg metrics in dimension $n \geq 3$.

Theorem 1.7. (Li-Shen [17]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an almost regular non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Then F is a weakly Landsberg metric if and only if β satisfies*

$$r_{ij} = k(b^2 a_{ij} - b_i b_j), \qquad s_{ij} = 0,$$

where $k = k(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi(s) := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1+sQ)Q'' = \frac{\lambda}{\sqrt{b^2-s^2}}\Delta^{\frac{3}{2}}, \tag{1.4}$$

where λ is a constant.

In [9], Chen-Liu proved that every regular (α,β)-metric $F = \alpha\phi(\beta/\alpha)$ satisfying (1.4) is Riemannian.

In [44], Zou-Cheng called the weak Landsberg metrics that are not of Berwald type the generalized unicorns. They studied generalized unicorn problem on regular (α,β)-metrics and proved the following.

Theorem 1.8. (Zou-Cheng [44]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi = \phi(s)$ is a polynomial in s. Then F is a weakly Landsberg metric if and only if it is a Berwald metric.*

In [43], Zhou-Wang-Li classified almost regular Landsberg general (α,β)-metrics into three cases and prove that those regular metrics must be Berwald metrics. By solving some nonlinear PDEs, some new almost regular Landsberg metrics are constructed which have not been described before.

In [6], Berwald gave the definition of stretch curvature as a generalization of Landsberg curvature. He showed that a Finsler metric is a stretch metric if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Then, this curvature investigated by Shibata in [25] and Matsumoto in [20]. In [35], Tayebi-Sadeghi characterized the class of stretch (α,β)-metrics with vanishing S-curvature.

Theorem 1.9. (Tayebi-Sadeghi [35]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α, β)-metric with vanishing S-curvature on a manifold M of dimension $n \geq 3$. Suppose that F is a stretch metric. Then one of the following holds:*

(i) If F is a regular metric, then it reduces to a Berwald metric;

(ii) If F is an almost regular metric which is not Berwaldian, then ϕ is given by (1.2). In this case, F is not a Landsberg metric.

There is a weaker notion of stretch metrics, the so-called weakly stretch metrics. Taking trace with respect to $\mathbf{g}_y$ in first and second variables of stretch curvature $\mathbf{\Sigma}_y$ gives rise mean stretch curvature $\bar{\mathbf{\Sigma}}_y$. A Finsler metric with vanishing mean stretch curvature is called a weakly stretch metric. For more information about the stretch and weakly stretch metrics, see [22] and [35]. By definition, we have the following:

$$\{\text{Landsberg metrics}\} \subseteq \{\text{Weakly Landsberg metrics}\} \subseteq \{\text{Weakly stretch metrics}\}.$$

In this paper, the weakly stretch metrics that are not of weakly Landsberg type are called the weakly generalized unicorns. In order to find weakly generalized unicorns, we consider the class of weakly stretch (α, β)-metrics. Then, we prove the following.

Theorem 1.10. (Tayebi-Najafi [32]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α, β)-metric on a manifold M, i.e., $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ for any constants $c_1 > 0$, c_2 and c_3. Suppose that F is a weakly stretch metric with vanishing S-curvature. Then one of the following hold:*

(i) If F is a regular metric, then it reduces to a Berwald metric;

(ii) If F is an almost regular metric which is not Berwaldian, then $\phi = \phi(s)$ is given by

$$\phi(s) = c\exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}}dt\right],$$

where $c > 0$, $q > 0$ and k are real constants. In this case, F is not a weakly Landsberg metric. More precisely, F is an almost regular weakly generalized unicorn.

Using the $\mathbf{S}$-curvature, one can define the non-Riemannian curvatures $\mathbf{\Xi} = \Xi_i dx^i$ and $\mathbf{H} = H_{ij}dx^i \otimes dx^j$ on the pullback tangent bundle π^*TM as follows

$$\Xi_i := \mathbf{S}_{.i|m}y^m - \mathbf{S}_{|i},$$
$$H_{ij} := \frac{1}{2}\mathbf{S}_{.i.j|m}y^m,$$

where $\mathbf{S}$ denotes the $\mathbf{S}$-curvature and "." and "|" denote the vertical and horizontal covariant derivatives, respectively, with respect to the Berwald connection of F. In [36], Tayebi-Tabatabaeifar studied the (α, β)-metrics defined by (1.2) and proved the following.

Theorem 1.11. (Tayebi-Tabatabaeifar [36]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M defined by following*

$$\phi(s) = \exp\left[\int_0^s \frac{kt + q\sqrt{b^2 - t^2}}{1 + kt^2 + qt\sqrt{b^2 - t^2}}dt\right],$$

where $q > 0$ and k are real constants such that $q \neq \sqrt{3}k$. Suppose that β satisfies following

$$r_{ij} = c(b^2 a_{ij} - b_i b_j), \qquad s_{ij} = 0,$$

where $c = c(x)$ is a scalar function on M. Then, the following are equivalent

(i) F has almost vanishing $\mathbf{H}$-curvature $\mathbf{H} = (n+1)/2F^{-1}\theta\ \mathbf{h}$;

(ii) F has almost vanishing $\mathbf{\Xi}$-curvature $\Xi_i = -(n+1)(\theta_i F - F_{y^i}\theta)$,

where $\theta := \theta_i(x)y^i$ is a 1-form on M and $\boldsymbol{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric. Moreover if (i) or (ii) holds, then F reduces to a Berwald metric.

2. Unicorns in Conformal Geometry

Independently of Asanov's works, the unicorn problem have appeared since 2001 in Matsumoto's study of conformal rigidity, related to the invariance problem of the mixed Berwald curvature under conformal changes [21]. In [39], using some weakening of the conformal invariance of the mixed Berwald curvature, Vincze characterized unicorns with closed Finsleroid axis 1-forms as the solutions to a conformal rigidity problem. He investigated the existence of (non-Riemannian) Finsler metrics admitting a (non-homothetic) conformal change such that the mixed curvature tensor of the Berwald connection contracted by the derivatives of the logarithmic scale function is invariant. He proved that any solution must be locally conformal to a unicorn with closed Finsleroid axis 1-form. Conversely, a unicorn with closed Finsleroid axis 1-form admits a local conformal change satisfying the rigidity condition.

In [12], Elgendi studied the unicorn problem from the conformal transformation point of view. Let $F = F(x, y)$ and $\tilde{F} = \tilde{F}(x, y)$ be two Finsler metrics on a manifold M. Then F is conformal to $\tilde{F}$ if and only if there exists a scalar function $\kappa = \kappa(x)$ on M such that $F(x, y) = e^{\kappa(x)}\tilde{F}(x, y)$. The scalar function $\kappa = \kappa(x)$ is called the conformal factor. It is interesting to find the conformal transformation of Landsberg or Berwald metrics which can produce a regular unicorn. By the help of [12], one can conclude that, without loss of generality, the non-regular examples of unicorns can be chosen in such a way that it is a conformal transformation of a Berwald or Minkowski metrics by a function $f(x^1)$ and hence $\beta = f(x^1)y^1$, i.e., $b_1 = f(x^1)$. Consequently, the directions of singularities of the metric will be $(\pm 1, 0, ..., 0)$. So, if the Finsler function has extreme directions in the directions of y^1, say, then β can be in the form $\beta = f(x^1)y^1$. For example, let $M = \mathbb{R}^3$ and

$$\alpha = \sqrt{(y^1)^2 + e^{2x^1}\left[(y^2)^2 + (y^3)^2\right]}, \qquad \beta = y^1.$$

In [24], it is proved the following metric is an unicorn

$$F = \sqrt{\alpha^2 + \beta\sqrt{\alpha^2 - \beta^2}}\, e^{\frac{1}{\sqrt{3}}\ \arctan\left(\frac{2\beta}{\sqrt{3(\alpha^2-\beta^2)}} + \frac{1}{\sqrt{3}}\right)}. \tag{2.1}$$

One can obtain the example by applying the strategy mentioned in [12]. If we choice

$$\alpha = \sqrt{e^{-2x^1}(x^1)^2 + ((y^2)^2 + (y^3)^2)}, \qquad \beta = e^{-x^1}y^1$$

then the Finsler metric (2.1) reduces to a Berwald metric. Now, applying the conformal transformation on F by the function e^{x^1} we will get the same metric. Moreover, making use of [12], the conformal transformation of F by any positive smooth function $f(x^1)$ will yield a unicorn. In [11], Elgendi proved the following.

Theorem 2.1. (Elgendi [11]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on a manifold M of dimension $n \geq 3$ which is defined by following*

$$F = \left(a\beta + \sqrt{\alpha^2 - \beta^2}\right) e^{\frac{a\beta}{a\beta + \sqrt{\alpha^2-\beta^2}}}, \tag{2.2}$$

where α and β are given by $\alpha = f(x^1)\sqrt{(y^1)^2 + \varphi(\hat{y})}$ and $\beta = f(x^1)y^1$, respectively. Here $a \neq 0$ is a constant, $f = f(x^1)$ is a positive function on $\mathbb{R}$ and φ is arbitrary quadratic function in $\hat{y}$ and $\hat{y}$ stands for the variables $y^2, \cdots, y^n$. The function φ should be chosen in such a way the metric tensor of α is non-degenerate. Then the Finsler metric (2.2) is a unicorn.

3. Unicorns in Projective Geometry

In [41], Zheng-He find the necessary and sufficient conditions under which an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is projectively related to a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, provided that ϕ is not given by (1.2). More precisely, they proved the following.

Theorem 3.1. (Zheng-He [41]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric and $\bar{F} = \bar{\alpha} + \bar{\beta}$ a Randers metric on a manifold M of dimensional $n \geq 3$. Suppose that β is not parallel and that ϕ is not given by following*

$$\phi(s) = c_3 \exp\left[\int_0^s \frac{c_1\sqrt{1 - (t/b_0)^2} + c_2 t}{1 + t(c_1\sqrt{1 - (t/b_0)^2} + c_2 t)}\, dt\right].$$

Then F is projectively related to $\bar{F}$ if and only if the following hold:

$$\begin{aligned}
&\big(1+(k_1+k_2s^2)s^2+k_3s^2\big)\phi''=(k_1+k_2s^2)(\phi-s\phi'),\\
&b_{i|j}=2\tau\big((1+k_1b^2)a_{ij}+(k_2b^2+k_3)b_ib_j\big),\\
&G^i_\alpha=G^i_{\bar{\alpha}}+\theta y^i-\tau(k_1\alpha^2+k_2\beta^2)b^i,\\
&d\bar{\beta}=0,
\end{aligned}$$

where $\tau=\tau(x)$ is a scalar function and $\theta=\theta_i(x)y^i$ is a 1-form on M, and k_1, k_2 and k_3 are constants with $(k_2,k_3)\neq(0,0)$.

Recently, Chen-Liu studied the unicorn problem in the set of almost regular Douglas (α,β)-metrics and proved the following.

Theorem 3.2. (Chen-Liu [8]) *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an non-Riemannian almost regular Douglas (α,β)-metric on a manifold M of dimension $n\geq 3$. Then F is weakly Landsberg metric if and only if it is Berwald metric.*

Since every Landsberg metric is a weakly Langsberg metric, then Chen-Liu's theorem implies that there is not unicorn in the set of almost regular Douglas (α,β)-metrics.

There is an important projective invariant in Finsler geometry, namely generalized Douglas-Weyl metrics, that their Douglas curvatures satisfying

$$D^m_{ijk|s}y^s=T_{ijk}y^m$$

for some tensor T_{jkl}, where "|" denotes the horizontal covariant derivatives with respect to the Berwald connection of F. This equation is equivalent to that for any parallel vector fields $u=u(t)$, $v=v(t)$ and $w=w(t)$ along a geodesic $c(t)$, there is a function $T=T(t)$ such that

$$\frac{d}{dt}\big[D_{\dot{c}}(u,v,w)\big]=T\dot{c}.$$

The geometric meaning of this identity is that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic [34].

Theorem 3.3. (Tayebi-Sadeghi [34]) *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be a non-Randers type (α,β)-metric on a manifold M. Then F is a generalized Douglas-Weyl metric with vanishing S-curvature if and only if one of the following holds:*

(i) If F is a regular metric, then it reduces to a Berwald metric;

(ii) If F is an almost regular metric which is not Berwaldian, then ϕ is given by

$$\phi=c\exp\left[\int_0^s\frac{kt+q\sqrt{b^2-t^2}}{1+kt^2+qt\sqrt{b^2-t^2}}dt\right],$$

where $c>0$, $q>0$ and k are real constants. In this case, F is not a Douglas metric nor Weyl metric.

As we mentioned for Theorem 2.1, in order to constructing new unicorns, Elgendi considered the conformal transformation of unicorns in [12]. In [29], Szilasi-Vincze gave an intrinsic proof of the Weyl theorem, which states that the projective and conformal properties of a Finsler metric determine its metric properties uniquely. Therefore the projective properties of Finsler metrics deserve extra attention.

In order to find new unicorns, one can consider β-change of a unicorn. Transformations or changes of Finsler metrics have a lot of applications not only in Differential Geometry but also in Physics. Let (M,F) be a Finsler manifold and $\beta=b_i(x)y^i$ be a one-form on M. In [18], Matsumoto introduced the special transformation of Finsler metric, namely Randers change of F, which is defined by $\tilde{F}:=F+\beta$. In [14], Hashiguchi-Ichijyō showed that the Randers change of F is projectively related to it if and only if β is closed with respect to F. Inspired of Randers change, Shibata introduced β-change of Finsler metrics and studied some geometrical properties of tensors being invariant by β-change of the metrics [26]. If a Finsler metric $\tilde{F}$ on M is given by $\tilde{F}=f(F,\beta)$, where $f=f(u,v)$ is a positively homogeneous function of degree 1 in F and β with certain smoothness conditions, then we say $\tilde{F}$ is a β-change of F. Moreover, if $\tilde{F}$ and F are projectively related, then we say the β-change is projective. In this paper, we prove the following.

Theorem 3.4. (Tayebi-Najafi [32]) *Let $F_U = F_U(x, y)$ be a unicorn on a manifold M of dimension $n \geq 3$. Then, every non-Randers projective β-change $\tilde{F} = f(F_U, \beta)$ of F_U is also an unicorn.*

Theorem 3.4 explains to the unicorn hunters that they should not expect to see such a creature in the jungle of non-Randers projective β-changes of unicorn (α, β)-metrics.

4. Unicorns in Homogeneous Spaces

In [40], Xu-Deng introduced a generalization of (α, β)-metrics, called (α_1, α_2)-metrics. Let (M, α) be an n-dimensional Riemannian manifold. Then one can define an α-orthogonal decomposition of the tangent bundle by $TM = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are two linear subbundles with dimensions n_1 and n_2 respectively, and $\alpha_i = \alpha|_{\mathcal{V}_i}$ $i = 1, 2$ are naturally viewed as functions on TM. An (α_1, α_2)-metric on M is a Finsler metric F which can be written as

$$F = \sqrt{L(\alpha_1^2, \alpha_2^2)}.$$

An (α_1, α_2)-metric can also be represented as

$$F = \alpha\phi\Big(\frac{\alpha_2}{\alpha}\Big) = \alpha\psi\Big(\frac{\alpha_1}{\alpha}\Big)$$

in which $\phi(s) = \psi(\sqrt{1 - s^2})$. They proved the following.

Theorem 4.1. (Xu-Deng [40]) *Any Landsberg (α_1, α_2)-metric is a Berwald metric.*

This result shows that the finding a unicorn cannot be successful even in the very broad class of (α_1, α_2)-metrics. Then, Xu-Deng conjectured the following:

Conjecture 4.2. ([40]) *A homogeneous Landsberg space must be a Berwald space.*

A Finsler space (M, F) is called homogeneous Finsler space if the group of isometries of (M, F) acts transitively on M. But in [31], Tayebi-Najafi gave a little toehold to the Deng-Xu's conjecture and proved the following.

Theorem 4.3. (Tayebi-Najafi [31]) *A homogeneous (α, β)-metric is a stretch metric if and only if it is a Berwald metric.*

Taking a look at the rigid theorems in Finsler geometry, one can find that this type of result is different for procedures with dimensions greater than three. For example, in [28] Szabó proved that any connected Berwald surface is locally Minkowskian or Riemannian. In [4], Bao-Chern-Shen proved a rigidity result for compact Landsberg surface. They showed that a compact Landsberg surfaces with non-positive flag curvature is locally Minkowskian or Riemannian. Therefore, we preferred to consider the issue of unicorns for homogeneous Finsler surfaces. In [33], Tayebi-Najafi proved the following rigidity result.

Theorem 4.4. ([33]) *Any homogeneous Landsberg surface is Riemannian or locally Minkowskian.*

This result articulates the hunters of unicorns that they do not looking forward to seeing such a creature in the jungle of homogeneous Finsler surfaces.

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