



Original Article

## On Finsler metrics with weakly isotropic $S$ -curvature

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**ABSTRACT:** In this paper, we focus on a class of Finsler metrics which are called general  $(\alpha, \beta)$ -metrics:  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a Riemannian metric and  $\beta = b_i(x)y^i$  is a 1-form. We examine the metrics as weakly isotropic  $S$ -curvature.

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(Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration)

## 1. Introduction

The  $S$ -curvature plays an essential role in Finsler geometry. It has been introduced by Z. Shen while he was studying on the volume form in Finsler geometry, [6]. Therefore, many authors have studied on this idea and obtained some important results, [9], [8], [10] and [11]. A Finsler metric of an isotropic  $S$ -curvature is defined as follows:

$$\mathbf{S} = (n + 1)cF, \quad (1.1)$$

$c = c(x)$  is a scalar function on  $M$ .

The  $E$ -curvature  $\mathbf{E} = E_{ij}dx^i \otimes dx^j$  is another Riemannian quantity which has been obtained from the  $S$ -curvature. In fact, it is introduced as follows:

$$E_{ij} = \frac{1}{2} \frac{\partial^2 S}{\partial y^i \partial y^j}. \quad (1.2)$$

A Finsler metric  $F$  of an isotropic  $E$ -curvature defined as follows: there is a scalar function  $c = c(x)$  on  $M$  such that

$$\mathbf{E} = \frac{1}{2}(n + 1)cF^{-1}\mathfrak{h}, \quad (1.3)$$

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$\mathfrak{h}$  is a family of bilinear forms  $\mathfrak{h}_y = \mathfrak{h}_{ij}dx^i \otimes dx^j$ , which are defined by  $\mathfrak{h}_{ij} := FF_{y^i y^j}$ .

By (1.2), one can easily realize that Finsler metric of isotropic  $E$ -curvature is of isotropic  $S$ -curvature. However, the converse is still an open problem. In [1], Cheng-Shen have proved that (1.1) is equivalent to (1.3) for Randers metrics. Then, X. Chun-Huan, X. Cheng, I.Y. Lee and M.H. Lee have obtained a similar result for some special  $(\alpha, \beta)$ -metrics [13], [4]. Najafi-Tayebi have obtained a condition on  $(\alpha, \beta)$ -metrics which has been verified that (1.1) and (1.3) are equivalent [5]. All these studying inspire us to focus on the idea for the general  $(\alpha, \beta)$ -metrics. There are some progress and results on the general  $(\alpha, \beta)$ -metrics, (see, [17], [19], [20]). The general  $(\alpha, \beta)$ -metrics has been introduced by C. Yu and H. Zhu, [15]. These class of metrics are defined as follow:

$$F = \alpha\phi(b^2, s),$$

$\alpha := \sqrt{a_{ij}(x)y^i y^j}$  and  $\beta := b_i(x)y^i$  ( $b := \|\beta\|_\alpha$ ) are Riemannian metric and 1-form, respectively. Here, also  $\phi(b^2, s)$  is a positive smooth function. In 2011, Yu and Zhu have obtained a sufficient condition for general  $(\alpha, \beta)$ -metrics to be locally projectively flat [15]. Then, they have completely classified the general  $(\alpha, \beta)$ -metrics with constant flag curvature under some suitable conditions. Moreover, They have constructed many new projectively flat Finsler metrics that these metrics are of constant flag curvature which are 1, 0 and  $-1$ , [16]. Many authors have obtained some important result and classification on the general  $(\alpha, \beta)$  - metrics, (see, [2], [3], [12], [14], [18], [21]).

For a general  $(\alpha, \beta)$  - metric, we use

$$\begin{aligned} Q &= \frac{\phi_2}{\phi - s\phi_2}, \\ \Delta &= 1 + sQ + (b^2 - s^2)Q_2, \\ \Phi &= -(Q - sQ_2)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q_{22}, \end{aligned}$$

and

$$\Xi = (s + b^2Q) \frac{\Phi}{\Delta^2}.$$

Focusing on the method in [5, 20], we study the general  $(\alpha, \beta)$ -metric to be a weakly isotropic  $S$ -curvature.

We give the following theorem:

**Theorem 1.1.** *Let  $F = \alpha\phi(b^2, s)$  be a general  $(\alpha, \beta)$ -metric on  $M^n$ . Suppose that  $\Xi$  and  $b$  are not constant.  $F$  is to be of a weakly isotropic  $S$ -curvature if and only if  $F$  is to be of an isotropic  $S$ -curvature.*

B. Najafi and A. Tayebi have proved that if  $F$  is an  $(\alpha, \beta)$  - metrics of isotropic  $S$ -curvature, then  $b$  is a constant term, [5]. However, If  $F$  is a general  $(\alpha, \beta)$  metric, then  $b$  is not necessarily to be a constant term. Moreover, if  $b$  is a constant term, then it has been obtained that the general  $(\alpha, \beta)$ -metrics have reduced to  $(\alpha, \beta)$ -metrics. According to these discussion, we suppose that  $b$  is not a constant term.

## 2. Preliminaries

$F$  be a Finsler metric on  $M^n$ . Every Finsler metric  $F$  induces a spray  $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ . The spray coefficients  $G^i$  are defined by

$$G^i := \frac{1}{4}g^{il} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \},$$

where  $g^{ij}$  is the inverse of the fundamental tensor  $g_{ij} := [\frac{1}{2}F^2]_{y^i y^j}$ . For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as  $G^i(x, y) = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ .

$S$  - curvature, is given by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} \left[ \ln \sigma_{BH} \right],$$

$dV_F = \sigma_F(x)dx^1 \wedge \dots \wedge dx^n$  is the Busemann-Hausdorff volume form.

$E$ -curvature  $\mathbf{E} = E_{ij}dx^i \otimes dx^j$  of  $F$  is defined by

$$E_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{\partial G^m}{\partial y^m} \right).$$

**Definition 2.1.** [7] Let  $F$  be a Finsler metric on  $M^n$ . Then

(a)  $F$  is to be a weakly isotropic  $S$ -curvature if  $\mathbf{S} = (n + 1)\mathbf{c}F + \eta$ ,

(b)  $F$  is to be a isotropic  $S$ -curvature if  $\mathbf{S} = (n + 1)\mathbf{c}F$ ,

$\mathbf{c} = \mathbf{c}(x)$  is a scalar function on  $M$ ,  $\eta = \eta_i(x)y^i$  is a 1 - form on  $M$ .

It is obvious that if  $F$  is of isotropic  $E$ -curvature iff  $F$  is of weakly isotropic  $S$ -curvature.

We introduce the well known identities as follows:

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{00} = r_{ij}y^i y^j, \quad s^i_0 = a^{ij}s_{jk}y^k,$$

$$r_i = b^j r_{ji}, \quad s_i = b^j s_{ji}, \quad r_0 = r_i y^i, \quad s_0 = s_i y^i, \quad r^i = a^{ij}r_j, \quad s^i = a^{ij}s_j, \quad r = b^i r_i,$$

where  $(a^{ij}) = (a_{ij})^{-1}$  and  $b^i := a^{ij}b_j$ .

To prove the main theorem, we give some essential facts given below:

**Lemma 2.2.** [15] The spray coefficients  $G^i$  of a general  $(\alpha, \beta)$ -metric  $F = \alpha\phi(b^2, s)$  are related to the spray coefficients  ${}^\alpha G^i$  of  $\alpha$  and given by

$$G^i = {}^\alpha G^i + \alpha Q s^i_0 + \{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\} \frac{y^i}{\alpha}$$

$$+ \{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\} b^i - \alpha^2 R(r^i + s^i),$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2}, \quad \Theta = \frac{Q - sQ_2}{2\Delta}, \quad \Psi = \frac{Q_2}{2\Delta},$$

$$\Pi = \frac{R_2 - 2sRQ_2 + sQR_2}{\Delta}, \quad \Omega = \frac{2R - sR_2 + 2b^2RQ_2 - b^2QR_2}{\Delta},$$

where  $\Delta := 1 + sQ + (b^2 - s^2)Q_2$ .

$S$ -curvature of general  $(\alpha, \beta)$ -metrics has been obtained by H. Zhu, [20]:

$$\mathbf{S} = (2\Psi + T - 2g)(r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q s_0) + \alpha Pr,$$

where

$$\Phi := -(Q - sQ_2)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q_{22}, \tag{2.1}$$

$$T := (n + 1)\Omega + s\Pi + \Pi_2(b^2 - s^2) - 2R + sR_2, \tag{2.2}$$

$$P := 2(n + 1)\Theta R + 4s\Psi R + 2(\Psi_2 R + \Psi R_2)(b^2 - s^2) + \Pi - R_2, \tag{2.3}$$

$g(b^2) := \frac{f'(b^2)}{f(b^2)}$ . Moreover, the classification of general  $(\alpha, \beta)$ -metric of isotropic  $S$ -curvature has given as follows, [20]:

**Lemma 2.3.** Let  $F = \alpha\phi(b^2, s)$  be a general  $(\alpha, \beta)$ -metric on  $M^n$ . Suppose that  $b$  is not a constant. Then  $F$  is of isotropic  $S$ -curvature if and only if one of the following satisfies

1)  $\phi$  satisfies

$$\frac{\Phi}{2\Delta^2} \mathfrak{d}(b^2 - s^2) + [(n + 1)E + H_2(b^2 - s^2) + 2sH - 2sg](\mathfrak{K} + \mathfrak{d}b^2) = (n + 1)\mathbf{c}\phi, \tag{2.4}$$

$$g(b^2) := \frac{f'(b^2)}{f(b^2)},$$

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} H,$$

$$H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\}}.$$

Moreover,  $\alpha$  and  $\beta$  satisfy

$$\begin{aligned} r_{ij} &= \mathfrak{K}a_{ij} + \mathfrak{D}b_i b_j, \\ s_i &= 0, \end{aligned}$$

where  $\mathfrak{K} = \mathfrak{K}(x)$ ,  $\mathfrak{D} = \mathfrak{D}(x)$  and  $\mathfrak{K} + \mathfrak{D}b^2 \neq 0$ .

2)  $\phi$  satisfies (2.4) and

$$(1 + \tau b^2)(2\Psi + T - 2g) - \frac{\Phi}{\Delta^2}(\tau s - Q) = 0.$$

Moreover,  $\alpha$  and  $\beta$  satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{D}b_i b_j + \tau(b_i s_j + b_j s_i),$$

where  $s_i \neq 0$  and  $1 + \tau b^2 \neq 0$ , where  $\tau = \tau(b^2)$ .

3)  $\phi$  satisfies (2.4) and

$$\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T - 2g) = 0.$$

Moreover,  $\alpha$  and  $\beta$  satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{D}b_i b_j + b_i \theta_j + b_j \theta_i, \quad s_i = 0,$$

where  $\theta = \theta_i(x)y^i \neq 0$  is a 1-form which is orthogonal to  $\beta$ .

**Lemma 2.4.** [20] A general  $(\alpha, \beta)$ -metric is a Riemannian metric if and only if  $\Phi = 0$ .

### 3. Weakly isotropic S-curvature

Firstly, we prove the following essential Lemma. The Lemma helps to prove the main theorem of this paper. We classify the general  $(\alpha, \beta)$ -metrics of weakly isotropic S-curvature, therefore we follow the following Lemma:

**Lemma 3.1.** Let  $F = \alpha\phi(b^2, s)$  be a general  $(\alpha, \beta)$ -metric  $M^n$ .  $F$  is of weakly isotropic S-curvature  $\mathbf{S} = (n + 1)\mathbf{c}F + \eta$  if and only if the following equality holds

$$\alpha^{-1} \frac{\Phi}{2\Delta^2} (r_{00} - 2\alpha Q s_0) - (2\Psi + T)(r_0 + s_0) - \alpha Pr = -(n + 1)\mathbf{c}F + \vartheta, \tag{3.1}$$

where

$$\vartheta := -2g(r_0 + s_0) - \eta,$$

and

$$g := \frac{f'(b^2)}{f(b^2)}.$$

**Proof.** Combining (2.1) and the definition of weakly isotropic S-curvature,  $\mathbf{S} = (n + 1)\mathbf{c}F + \eta$ , we prove the lemma.  $\square$

Simplifying (3.1), we need to use the special coordinate as follows:  $\psi : (s, u^a) \rightarrow (y^i)$  by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^a = u^a, \tag{3.2}$$

where

$$\bar{\alpha} = \sqrt{\sum_{a=2}^n (u^a)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

Take a special coordinate system at an arbitrary  $x$  as in (3.2). It is easy to get

$$r_1 = br_{11}, \quad r_a = br_{1a}, \quad r = b^2r_{11}, \quad s_1 = 0, \quad s_a = bs_{1a}.$$

Let

$$\begin{aligned} \bar{r}_{10} &:= \sum_{a=2}^n r_{1a}y^a, \quad \bar{r}_{00} := \sum_{a,b=2}^n r_{ab}y^ay^b, \quad \bar{r}_0 := \sum_{a=2}^n r_ay^a, \\ \bar{s}_{10} &:= \sum_{a=2}^n s_{1a}y^a, \quad \bar{s}_0 := \sum_{a=2}^n s_ay^a. \end{aligned}$$

Put

$$\vartheta = t_iy^i - \eta_iy^i.$$

Then  $t_i$  are given by

$$t_1 = -2bgr_{11}, \quad t_a = -2bg(r_{1a} + s_{1a}).$$

A direct computation yields

$$r_0 = r_{11} \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha} + b\bar{r}_{10}, \quad s_0 = \bar{s}_0 = b\bar{s}_{10},$$

and

$$\begin{aligned} r_{00} &= r_{11} \frac{s^2}{b^2 - s^2}\bar{\alpha}^2 + 2\bar{r}_{10} \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} + \bar{r}_{00}, \\ \vartheta &= -2bg \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha} - 2bg(\bar{r}_{10} + \bar{s}_{10}) - \eta. \end{aligned}$$

When we plug the expressions obtained above into (3.1), we verify that (3.1) is equivalent to the following equations:

$$\left\{ \left[ \frac{s^2\Phi}{2\Delta^2} - sb^2(2\Psi + T) - b^4P \right] r_{11} + (n+1)cb^2\phi - sbt_1 \right\} \bar{\alpha}^2 + \frac{\Phi}{2\Delta^2}(b^2 - s^2)\bar{r}_{00} = 0, \quad (3.3)$$

$$\left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) \right] (r_{1a} + s_{1a}) - (s + b^2Q) \frac{\Phi}{\Delta^2} s_{1a} + b\eta_a - bt_a = 0, \quad (3.4)$$

$$\eta_1 = 0. \quad (3.5)$$

Since  $F$  is a non-Riemannian metric,  $\Phi \neq 0$  by Lemma 2.4. It is obvious that  $\bar{r}_{00}$ , and  $\bar{\alpha}$  are independent of  $s$ . Following (3.3), and (3.4), we see that the following relations hold in a special coordinate system  $(s, y^a)$  at a point  $x$ :

$$r_{ab} = \mathfrak{K}\delta_{ab}, \quad (3.6)$$

$$\left[ \frac{s^2\Phi}{2\Delta^2} - sb^2(2\Psi + T) - b^4P \right] r_{11} + (n+1)cb^2\phi - sbt_1 + \frac{\mathfrak{K}\Phi}{2\Delta^2}(b^2 - s^2) = 0, \quad (3.7)$$

$$\left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) \right] (r_{1a} + s_{1a}) - (s + b^2Q) \frac{\Phi}{\Delta^2} s_{1a} + b\eta_a - bt_a = 0, \quad (3.8)$$

$\mathfrak{K} = \mathfrak{K}(x)$  is independent of  $s$ . (3.6) satisfies that there is a 1 - form  $\theta$ , [20]. Then, we have

$$r_{ij} = \mathfrak{K}(x)a_{ij} + \mathfrak{D}(x)b_ib_j + b_i\theta_j + b_j\theta_i, \quad (3.9)$$

for some scalar function  $\mathfrak{K}(x), \mathfrak{D}(x)$  and some 1 - form  $\theta$ . In fact, (3.6) is equivalent to

$$r_{00} = \mathfrak{K}(x)\alpha^2, \quad \forall y \in (\beta^\#)^\perp, \quad (3.10)$$

where  $(\beta^\#)^\perp := \{(y^i) \in T_x M \mid b_i y^i = 0\}$ . Notice that any vector lying in hyperplane  $\beta = 0$  can be represented as  $b^2 y^i - \beta b^i$ . Substituting it into (3.10), one can see that (3.9) holds. Also, we always assume that  $\theta$  is perpendicular to  $\beta$ , i.e.,  $\theta_i b^i = 0$ . That is because if  $\theta$  is not orthogonal to  $\beta$ , we can represent  $\theta$  as  $\theta' + \frac{\theta^i b_i}{b^2} \beta$ , therefore  $\theta'$  orthogonal to  $\beta$ .

By (3.9), we have

$$r_{11} = \mathfrak{K} + \mathfrak{d}b^2, \quad r_{1a} = b\theta_a. \tag{3.11}$$

Plugging (3.11) into (3.7) and (3.8) yields

$$\left[ \frac{s^2 \Phi}{2\Delta^2} - sb^2(2\Psi + T) - b^4 P + 2sb^2 g \right] (\mathfrak{K} + \mathfrak{d}b^2) + (n+1)c b^2 \phi + \frac{\mathfrak{K}\Phi}{2\Delta^2} (b^2 - s^2) = 0, \tag{3.12}$$

$$\left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) + 2b^2 g \right] (b^2 \theta_a + s_a) - (s + b^2 Q) \frac{\Phi}{\Delta^2} s_a + b^2 \eta_a = 0. \tag{3.13}$$

We state the the following Proposition:

**Proposition 3.2.** *Let  $F = \alpha\phi(b^2, s)$  be a non-Riemannian general  $(\alpha, \beta)$ -metric on  $M^n$ . Suppose that  $b$  is not a constant.  $F$  is to be a weakly isotropic  $S$ -curvature if and only if one of the following holds*

1)  $\phi$  satisfies

$$\frac{\Phi}{2\Delta^2} \mathfrak{d}(b^2 - s^2) + [(n+1)E + H_2(b^2 - s^2) + 2sH - 2sg] (\mathfrak{K} + \mathfrak{d}b^2) = (n+1)c\phi, \tag{3.14}$$

where  $g(b^2) := \frac{f'(b^2)}{f(b^2)}$ ,

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} H, \tag{3.15}$$

$$H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\}}. \tag{3.16}$$

Moreover,  $\alpha$  and  $\beta$  satisfy

$$\begin{aligned} r_{ij} &= \mathfrak{K}a_{ij} + \mathfrak{d}b_i b_j, \\ s_i &= 0, \end{aligned}$$

where  $\mathfrak{K} = \mathfrak{K}(x)$ ,  $\mathfrak{d} = \mathfrak{d}(x)$  and  $\mathfrak{K} + \mathfrak{d}b^2 \neq 0$ .

In that case,  $\mathbf{S} = (n+1)c\phi$ : that is,  $F$  is of isotropic  $S$ -curvature.

2)  $\phi$  satisfies (3.14) and

$$(1 + \tau b^2)(2\Psi + T - 2g) - \frac{\Phi}{\Delta^2} (\tau s - Q) = \frac{\eta_a}{s_a}.$$

Moreover,  $\alpha$  and  $\beta$  satisfy

$$r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_i b_j + \tau(b_i s_j + b_j s_i), \tag{3.17}$$

where  $s_i \neq 0$  and  $1 + \tau b^2 \neq 0$ , where  $\tau = \tau(b^2)$  and  $\eta = \eta_i(x)y^i$  is a 1 - form on  $M$ .

3)  $\phi$  satisfies (3.14) and

$$\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T - 2g) = -\frac{\eta_a}{\theta_a}. \tag{3.18}$$

Moreover,  $\alpha$  and  $\beta$  satisfy

$$\begin{aligned} r_{ij} &= \mathfrak{K}a_{ij} + \mathfrak{d}b_i b_j + b_i \theta_j + b_j \theta_i, \\ s_i &= 0, \end{aligned}$$

where  $\theta = \theta_i(x)y^i \neq 0$  is a 1 - form which is orthogonal to  $\beta$ , and  $\eta = \eta_i(x)y^i$  is a 1 - form on  $M$ .

**Proof.** Mainly, the sufficient part of the Proposition follows the proof of Proposition 4.1 in [20]. Thus we omit it. Hence, we just need to prove the necessary part. Suppose that  $F$  is of weak isotropic  $S$ -curvature, then (3.9), (3.12) and (3.13) hold. (3.12) is equivalent to the following

$$\frac{\Phi}{2\Delta^2} \mathfrak{d}(b^2 - s^2) + [(n + 1)E + H_2(b^2 - s^2) + 2sH - 2sg](\mathfrak{K} + \mathfrak{d}b^2) = (n + 1)\mathfrak{c}\phi,$$

where

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} H, \quad H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\}}.$$

Let us suppose that  $\Xi = (s + b^2Q) \frac{\Phi}{\Delta^2}$  is not constant. Since  $b$  is not constant, then we divide (3.9) into two cases:

I) If  $\theta = \tau(x)s_0$ , then according to  $b \neq \text{constant}$  we have three possible cases in the following

a)  $r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_i b_j$  and  $s_0 = 0$ , where  $\mathfrak{K} + \mathfrak{d}b^2 \neq 0$ .

In this case,  $\theta_a = 0$  and  $s_a = 0$ . It is easy to see that (3.13) is reduced to  $b^2\eta_a = 0$ . Hence, we have  $\eta_a = 0$ . Thus, by (3.5), we get  $\eta = 0$  and as a result we have that  $F$  is of isotropic  $S$ -curvature  $\mathbf{S} = (n + 1)\mathfrak{c}F$ .

b)  $r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_i b_j + \tau(b_i s_j + b_j s_i)$  and  $s_0 \neq 0$ , where  $1 + \tau b^2 \neq 0$ . In that case,  $\theta_a = \tau s_a$  and  $s_a \neq 0$ . (3.13) is reduced to

$$s_a \left\{ (1 + b^2\tau) \left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) + 2b^2g \right] - (s + b^2Q) \frac{\Phi}{\Delta^2} \right\} + b^2\eta_a = 0,$$

which is equivalent to (3.17).

c)  $r_{ij} = \mathfrak{K}a_{ij} + \mathfrak{d}b_i b_j - \frac{1}{b^2}(b_i s_j + b_j s_i)$  and  $s_0 \neq 0$ , where  $\mathfrak{K} + \mathfrak{d}b^2 \neq 0$ . In this case,  $\theta_a = -\frac{1}{b^2}s_a$  and  $s_a \neq 0$ . (3.13) is reduced to

$$-(s + b^2Q) \frac{\Phi}{\Delta^2} s_a + b^2\eta_a = 0.$$

In fact, (3.19) implies

$$\frac{\Xi}{b^2} s_a = \eta_a. \tag{3.19}$$

Since  $s_a \neq 0$ , using the last equation, we obtain that  $\frac{\Xi}{b^2}$  is a constant. It is a contradiction. This implies that We need to omit this case.

II)  $\theta \neq \tau(x)s_0$ .

In this case,  $r_{ij} = \mathfrak{K}(x)a_{ij} + \mathfrak{d}(x)b_i b_j + b_i \theta_j + b_j \theta_i$  and  $s_0 = 0$ , where  $\theta_i \neq 0$ . Hence,  $\theta_a \neq 0$  and  $s_a = 0$ . Therefore, (3.13) is reduced to

$$\left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) + 2b^2g \right] b^2\theta_a + b^2\eta_a = 0, \tag{3.20}$$

which is equivalent to (3.18). □

#### 4. The proof of Theorem 1.1

It is sufficient to prove that if  $F$  is of weakly isotropic  $S$ -curvature, then  $F$  is of isotropic  $S$ -curvature. In fact, it suffices to show that if (3.19) and (3.20) hold, then  $\eta_a = 0$ .

Firstly, Assume that (3.19) hold. We claim that  $\eta_a = 0$ . Let  $\eta_a \neq 0$ . By (3.19), we get

$$\frac{s_a}{b^2} \left\{ (1 + b^2\tau) \left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) + 2b^2g \right] - (s + b^2Q) \frac{\Phi}{\Delta^2} \right\} + \eta_a = 0, \tag{4.1}$$

Since  $s_a \neq 0$  and  $b$  is not a constant, by (4.1) it follows that

$$(1 + b^2\tau) \left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) + 2b^2g \right] - (s + b^2Q) \frac{\Phi}{\Delta^2} = 0. \tag{4.2}$$

By (4.1) and (4.2), we obtain  $\eta_a = 0$ .

Now, suppose that (3.20) hold. Let

$$\Upsilon := \left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) \right]_s.$$

We see that  $\Upsilon = 0$  if and only if

$$\frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) = b^2\mu,$$

where  $\mu = \mu(x)$  is independent of  $s$ .

$\Upsilon = 0$ :

By (3.20), we get

$$b^2[\mu + 2g]\theta_a + \eta_a = 0. \tag{4.3}$$

Since  $\theta_a \neq 0$  and  $b$  are not constant, it follows from (4.3) that

$$\mu + 2g = 0.$$

By (4.3), we have  $\eta_a = 0$ .

$\Upsilon \neq 0$ :

By (3.20), we get

$$\left[ \frac{s\Phi}{\Delta^2} - b^2(2\Psi + T) \right] \theta_a = -2b^2g\theta_a - \eta_a.$$

By the assumption  $\Upsilon \neq 0$ , it is fact that this is impossible. It is a contradiction. Hence  $\eta_a = 0$ .

□

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