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A note on the Yamabe problem of Randers metrics

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ABSTRACT: The classical Yamabe problem in Riemannian geometry states that every conformal class contains a metric with constant scalar curvature. In Finsler geometry, the C-convexity is needed in general. In this paper, we study the strong C-convexity of Randers metrics, and provide a result on the Yamabe problem for the metrics of Randers type.

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(*Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration*)

1. Introduction

H. Yamabe attempted to seek Riemannian metrics with constant scalar curvature in a conformal class in [11]. A bundle of works of N. Trudinger [10], T. Aubin [1] and R. Schoen [9] gives an affirmative answer to the Yamabe Problem, which is a milestone in Riemannian geometry. In Finsler realm, X. Cheng and M. Yuan [4] studied the Yamabe problem for the scalar curvature defined by H. Akbar-Zadeh, and obtained a negative answer for Randers metrics. In the view of calculus of variations, L. Zhao and the first author defined a Finsler scalar curvature $\mathrm{Scal}(x)$ and proved that *a Finsler metric with constant scalar curvature is a critical point of the total scalar curvature functional*

$$\mathcal{S}(F) = \frac{1}{\mathrm{Vol}(M)^{1-\frac{2}{n}}} \int_M \mathrm{Scal}(x)\, d\mu_F$$

in its conformal class([5]). The Yamabe invariant is defined as $Y(M,F) = \inf_u \mathcal{S}(e^{u(x)}F)$. In order to have a lower bound of $\mathcal{S}$ in the conformal class $[F]$ of the metric F, the condition *C-convex* is introduced in [5] which is conformally invariant. By introducing another conformal invariant $C(M,F)$, L. Zhao and the first author partially solved the Yamabe problem in Finsler geometry.

Theorem 1.1 ([5]). *Let (M^n, F) be a compact C-convex Finsler manifold with $n \geq 3$. If $Y(M,F)C(M,F) < Y(\mathbb{S}^n)$, then there exists a metric $\bar{F}$ conformal to F such that* $\mathrm{Scal}_{\bar{F}}(x) = Y(M,\bar{F})$.

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In [6], L. Zhao and the first author introduce the notion of *strongly C-convexity* (see (2.1) in §2) which is a bit more stronger than C-convexity. In this paper, we shall study the strong C-convexity of Randers metrics and obtain the following result.

Theorem 1.2. *A Randers metric $\alpha + \beta$ is strongly C-convex if and only if*

$$\|\beta\|_\alpha < B_n := \left(1 - \left(\frac{8n^3 - 16n^2 + 8n + 2}{n^4 + 3n^3 - 9n^2 + 7n}\right)^2\right)^{1/2}.$$

where the dimension $n \geq 3$.

As a conclusion, we can give a statement to the Yamabe Problem of Randers metrics.

Corollary 1.3. *Let $F = \alpha + \beta$ be a Randers metric on a compact manifold M^n with $n \geq 3$. If $\|\beta\|_\alpha < B_n$ and $Y(M,F)C(M,F) < Y(\mathbb{S}^n)$, then there exists a metric $\bar{F}$ conformal to F such that* $\mathrm{Scal}_{\bar{F}}(x) = Y(M, \bar{F})$.

The contents of this paper are arranged as follows. In §2, we give a brief review of Finsler metrics and give the precise definition of strongly C-convex. In §3, we study the strongly C-convexity of Randers metrics. Throughout this paper, we always assume that the dimension $n \geq 3$.

2. Finsler metrics

Let M be an n-dimensional differentiable manifold with $n \geq 3$. The points in the tangent bundle TM are denoted by (x, y), where $x \in M$ and $y \in T_xM$. Let $(x^i; y^i)$ be the local coordinates of TM with $y = y^i\partial/\partial x^i$.

Let $F : TM \to [0, +\infty)$ be a Finsler metric on M. The fundamental form of F is

$$g = g_{ik}(x,y)dx^i \otimes dx^k, \quad g_{ik} := \left[\frac{1}{2}F^2\right]_{y^iy^k}.$$

Here and from now on, the lower index x^i, y^i always means partial derivatives, such as $F_{y^i} := \frac{\partial F}{\partial y^i}$, $F_{x^i} := \frac{\partial F}{\partial x^i}$, $[F^2]_{y^iy^k} := \frac{\partial^2 F^2}{\partial y^i \partial y^k}$, and etc.

The canonical projection $\pi : TM\backslash\{0\} \to M$ gives rise to a covector bundle π^*T^*M, on which there exists the Hilbert form $\omega = \ell_i dx^i$ where $\ell_i = F_{y^i}$, whose dual is the distinguished section of π^*TM

$$\ell = \ell^i \frac{\partial}{\partial x^i}, \quad \text{with} \quad \ell^i := \frac{y^i}{F}.$$

The Cartan tensor (Cartan torsion) and the Cartan form are respectively

$$A = A_{ijk}dx^i \otimes dx^j \otimes dx^k, \quad A_{ijk} := \frac{F}{4}\left[F^2\right]_{y^iy^jy^k},$$

$$I = I_i dx^i, \quad I_i := A_{ijk}g^{jk}, \quad (g^{jk}) = (g_{jk})^{-1}.$$

The spray coefficients are given as

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\}$$

which determine the Berwald connection coefficients in the following way

$$\Gamma^i_{jk} = G^i_{y^jy^k}.$$

The flag curvature tensor (Riemann curvature tensor) is given by

$$R^i{}_k = 2G^i_{x^k} - G^i_{x^jy^k}y^j + 2G^jG^i_{y^jy^k} - G^i_{y^j}G^j_{y^k},$$

while the Ricci curvature is defined as the trace

$$Ric(x,y) := \frac{1}{F^2}R^i{}_i.$$

The most important non-Riemannian curvature in Finsler geometry is the Landsberg curvature, which is defined as the derivative of the Cartan torsion

$$L_{ijk} := A_{ijk:m}\ell^m$$

where ":" is the horizontal covariant derivative with respect to the Berwald connection. The mean Landsberg tensor is

$$J = J_k dx^k, \quad J_k := g^{ij} L_{ijk}.$$

On the punctured bundle $TM\backslash\{0\}$, there is the Sasaki type metric $g_{ik}dx^i \otimes dx^k + g_{ik}\frac{\delta y^i}{F} \otimes \frac{\delta y^k}{F}$, which induces a Riemannian metric on the projective sphere bundle SM

$$\hat{g} = g_{ik}dx^i \otimes dx^k + F[F]_{y^i y^k}\frac{\delta y^i}{F} \otimes \frac{\delta y^k}{F}.$$

Hence the volume form of SM can be expressed as [3, 7]

$$d\mu_{SM} = \Omega d\eta \wedge dx, \quad \Omega := \det\left(\frac{g_{ik}}{F}\right)$$

where

$$d\eta := \sum (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n, \quad dx := dx^1 \wedge \cdots \wedge dx^n.$$

The volume form of M induced by SM can be defined by

$$d\mu_F = \sigma_F(x)dx, \quad \sigma_F(x) := \frac{1}{\omega_{n-1}} \int_{S_x M} \Omega d\eta,$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional standard sphere.

By integrating along the fibre, the *scalar curvature* can be defined as

$$\mathrm{Scal}(x) = \frac{n\displaystyle\int_{S_x M} Ric \cdot \Omega d\eta}{\displaystyle\int_{S_x M} \Omega d\eta} + \frac{2n}{n-2} \frac{\displaystyle\int_{S_x M} g^{ij} J_{i:j} \cdot \Omega d\eta}{\displaystyle\int_{S_x M} \Omega d\eta}$$

as the dimension $n \geq 3$. In order to obtain the existence of Finsler metrics with constant scalar curvature $\mathrm{Scal}(x)$, the concept of C-convexity is introduced in [5]. Precisely, a Finsler metric is *strongly C-convex* if the tensor

$$\mathfrak{C}^{ij} = g^{ij} - \frac{n}{(n-1)(n-2)}(\ell^i I^j + \ell^j I^i + A^{ir}_s A^{js}_r) \tag{2.1}$$

is positive definite, while *C-convexity* means the positivity of the tensor

$$c^{ij} = \frac{1}{\int_{S_x M} \Omega d\eta} \int_{S_x M} \mathfrak{C}^{ij} \cdot \Omega d\eta.$$

We shall point out that the C-convexity does not make sense as $n = 2$.

One can find that a metric is strongly C-convex if its Cartan torsion is sufficiently small. In the next section, we shall study the stongly C-convexity of Randers metrics and obtain Theorem 1.2.

3. Strongly C-convexity of Randers Metrics

Let $F = \alpha + \beta$ be a Randers metric where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ with $b = \sqrt{a^{ij}b_i b_j} < 1$ and $(a^{ij}) = (a_{kl})^{-1}$. In this section, we shall investigate the positivity of $(\mathfrak{C}^{ij})$ of $F = \alpha + \beta$.

It is well-known that the Cartan tensor of a Randers metric is reducible

$$A_{ijk} = \frac{1}{n+1}\{I_i h_{ik} + I_j h_{ik} + I_k h_{ij}\}$$

where $h_{ij}: = FF_{y^i y^j} = g_{ij} - \ell_i \ell_j$ is the angular tensor, and the Cantan form is

$$I_i = \frac{n+1}{2}\left(b_i - \frac{\beta}{\alpha}\alpha_{y^i}\right).$$

Being aware of $h^r_s h^s_r = n-1$ and $h^i_s h^{js} = g^{ij} - \ell^i \ell^j$, one can get

$$A^{ir}_s A^{js}_r = \frac{1}{(n+1)^2}\{2\|I\|^2 g^{ij} + (n+5)I^i I^j - 2\|I\|^2 \ell^i \ell^j\}$$

where

$$\|I\|^2 = I_i I_j g^{ij} = \frac{(n+1)^2}{4} \cdot \frac{b^2 - (\beta/\alpha)^2}{1 + (\beta/\alpha)} \leq \frac{(n+1)^2}{2} \left(1 - \sqrt{1-b^2}\right), \tag{3.1}$$

which can be found in [8]. Thus, we reach

$$\begin{aligned}\mathfrak{C}^{ij} = &\left(1 - 2\|I\|^2 \frac{n}{(n+1)^2(n-1)(n-2)}\right) g^{ij} \\ &- \frac{n}{(n-1)(n-2)} \left(\ell^i I^j + \ell^j I^i + \frac{n+5}{(n+1)^2} I^i I^j - \frac{2\|I\|^2}{(n+1)^2} \ell^i \ell^j\right).\end{aligned}$$

For investigating the positivity, one can apply the continuity method. Let us consider the family $F_t = \alpha + t\beta$ where $t \in [0, 1]$. It is clear that $(\mathfrak{C}^{ij})$ of F_0 is (a^{ij}) which is positive definite. Hence, once we obtain the invertibility of $(\mathfrak{C}^{ij})$ for every F_t, we shall have the positivity of $(\mathfrak{C}^{ij})$ for every F_t. Thus we shall calculate the determinant $\det(\mathfrak{C}^{ij})$ by applying the following lemma.

Lemma 3.1 ([2]). *Let $H = (H^{ij})$ be a symmetric $n \times n$ matrix and $V = (V^i)$ be an n-vector. Put $G^{ij} = H^{ij} + \delta V^i V^j$ where δ is a complex number. Assuming that H is invertible with $H^{-1} = (H_{ij})$, it holds*

$$\det(G^{ij}) = (1 + \delta v) \det(H^{ij}),$$

where $v = V_i V^i$ and $V_i = H_{ij} V^j$. Moreover, if $1 + \delta v \neq 0$, G is invertible and the inverse $G^{-1} = (G_{ij})$ is given by

$$G_{ij} = H_{ij} - \frac{\delta V_i V_j}{1 + \delta v}.$$

In order to apply the above lemma, we rewrite $\mathfrak{C}^{ij}$ in the following form

$$\mathfrak{C}^{ij} = \rho_0 g^{ij} + \rho_1 \ell^i \ell^j - \rho_2 \left(I^i + \frac{(n+1)^2}{n+5} \ell^i\right) \left(I^j + \frac{(n+1)^2}{n+5} \ell^j\right) \tag{3.2}$$

where

$$\begin{aligned}\rho_0 =& 1 - 2\|I\|^2 \frac{n}{(n+1)^2(n-1)(n-2)}, \\ \rho_1 =& \frac{n}{(n-1)(n-2)} \left(\frac{(n+1)^2}{n+5} + \frac{2\|I\|^2}{(n+1)^2}\right), \\ \rho_2 =& \frac{n(n+5)}{(n+1)^2(n-1)(n-2)}.\end{aligned}$$

Lemma 3.2. *The coefficient ρ_0 is positive when $n \geq 4$. In dimension $n = 3$, $\rho_0 > 0$ if and only if $b^2 < \frac{8}{9}$.*

Proof. By (3.1), we have $\|I\|^2 < \frac{(n+1)^2}{2}$, thus for $n \geq 4$ we have

$$\rho_0 = 1 - 2\|I\|^2 \frac{n}{(n+1)^2(n-1)(n-2)} > 1 - \frac{n}{(n-1)(n-2)} > 0.$$

For $n = 3$, we use the estimate $\|I\|^2 \leq \frac{(n+1)^2}{2}(1 - \sqrt{1-b^2})$ where the equality can be achieved. Thus $\rho_0 > 0$ if and only if

$$\min_y \rho_0 = 1 - \frac{3}{16} \max_y \|I\|^2 = 1 - \frac{3}{2}(1 - \sqrt{1-b^2}) > 0$$

which implies $b^2 < \frac{8}{9}$.

□

In the remaining part of this section, we shall assume $\rho_0 > 0$. Therefore, by putting

$$\widetilde{H}^{ij} = \rho_0 g^{ij} + \rho_1 \ell^i \ell^j = \rho_0 \left(g^{ij} + \frac{\rho_1}{\rho_0} \ell^i \ell^j\right),$$

and according to Lemma 3.1, we have

$$\det(\widetilde{H}^{ij}) = (\rho_0)^n \det(g^{ij}) \left(1 + \frac{\rho_1}{\rho_0}\right).$$

One can easily find that $1 + \frac{\rho_2}{\rho_0} > 0$. Thus $(\widetilde{H}^{ij})$ is invertible with the inverse

$$\widetilde{H}_{ij} = \frac{1}{\rho_0} g_{ij} - \frac{\rho_1}{(\rho_0)^2} \frac{\ell_i \ell_j}{1 + \frac{\rho_1}{\rho_0}}.$$

Now, applying Lemma 3.1 to

$$\mathfrak{C}^{ij} = \widetilde{H}^{ij} - \rho_2 \left(I^i + \frac{(n+1)^2}{n+5} \ell^i \right) \left(I^j + \frac{(n+1)^2}{n+5} \ell^j \right),$$

we obtain

$$\begin{aligned} \det(\mathfrak{C}^{ij}) =& \det(\widetilde{H}^{ij}) \left[1 - \rho_2 \widetilde{H}_{ij} \left(I^i + \frac{(n+1)^2}{n+5} \ell^i \right) \left(I^j + \frac{(n+1)^2}{n+5} \ell^j \right) \right] \\ =& (\rho_0)^n \det(g^{ij}) \left[1 - \frac{\rho_2}{\rho_0} \|I\|^2 - \frac{(n+1)^4}{(n+5)^2} \frac{\rho_2}{\rho_0 + \rho_1} \right] \\ =& (\rho_0)^n \det(g^{ij}) \left[1 - \frac{n(n+1)^2}{2n^3 + 4n^2 - 12n + 10} \right. \\ & \left. - \frac{n(n+5)\|I\|^2}{(n+1)^2(n-1)(n-2) - 2n\|I\|^2} \right]. \end{aligned}$$

Theorem 3.3. *A Randers metric F is strongly C-convex if and only if*

$$b < B_n := \left(1 - \left(\frac{8n^3 - 16n^2 + 8n + 2}{n^4 + 3n^3 - 9n^2 + 7n} \right)^2 \right)^{1/2}$$

where $b = \|\beta\|_\alpha$.

Proof. By the decomposition (3.2) and $n \geq 3$, the positivity of $(\mathfrak{C}^{ij})$ shall imply $\rho_0 > 0$. In fact, since $n \geq 3$, by picking a covector (V_i) such that

$$\ell^i V_i = 0, \quad \left(I^i + \frac{(n+1)^2}{n+5} \ell^i \right) V_i = 0,$$

we have $\mathfrak{C}^{ij} V_i V_j = \rho_0 g^{ij} V_i V_j$. Thus the positivity of $(\mathfrak{C}^{ij})$ does imply $\rho_0 > 0$. Hence, if F is strongly C-convex, we have $\rho_0 > 0$ and $\det(\mathfrak{C}^{ij}) > 0$. Therefore, one shall get

$$1 - \frac{n(n+1)^2}{2n^3 + 4n^2 - 12n + 10} - \frac{n(n+5)\|I\|^2}{(n+1)^2(n-1)(n-2) - 2n\|I\|^2} > 0. \tag{3.3}$$

The inequality (3.3) is equivalent to

$$\|I\|^2 < \frac{(n+1)^2(n-1)(n-2)(n(n-2)-1)}{2n(n^3 + 3n^2 - 9n + 7)}.$$

Since $\max_y \|I\|^2 = \frac{(n+1)^2}{2}(1 - \sqrt{1 - b^2})$, the above inequality holds if and only if

$$\frac{(n+1)^2}{2}(1 - \sqrt{1 - b^2}) < \frac{(n+1)^2(n-1)(n-2)(n(n-2)-1)}{2n(n^3 + 3n^2 - 9n + 7)},$$

from which we can get

$$b^2 < 1 - \left(\frac{8n^3 - 16n^2 + 8n + 2}{n^4 + 3n^3 - 9n^2 + 7n} \right)^2. \tag{3.4}$$

Conversely, let us assume that β satisfies (3.4). A simple calculation shows that $b^2 < \frac{8}{9}$ as $n = 3$. Hence, ρ_0 is positive according to Lemma 3.2. Since (3.15) is equivalent to (3.18), we have $\det(\mathfrak{C}^{ij}) > 0$ in this case. Now, put $F_t = \alpha + t\beta$ for $t \in [0, 1]$, and denote $\mathfrak{C}^{ij}$ of F_t by $\mathfrak{C}^{ij}(t)$. It is clear that $t\beta$ also satisfies (3.18) since $\|t\beta\|_\alpha \leq \|\beta\|_\alpha$. Hence, for every $t \in [0, 1]$ we have $\det(\mathfrak{C}^{ij}(t)) > 0$ and thus the eigenvalues of $\mathfrak{C}^{ij}(t)$ are nonzero. Note that the eigenvalues of $\mathfrak{C}^{ij}(t)$ depend continuously on t, and $\mathfrak{C}^{ij}(0) = a^{ij}$ is positive definite. As t changes from 0 to 1, none of these eigenvalues can become negative. Thus $\mathfrak{C}^{ij}$ is positive definite.

□

Remark. In order to have an intuition, we list below the decimal values of several B_n's.

	$n=3$	$n=4$	$n=5$	$n=10$	$n=100$	$n=1000$
B_n	0.2773	0.4869	0.6098	0.8464	0.9971	0.9999

As n grows, the condition on β becomes weaker. While it is critical as the dimension is low.

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