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Navigation problem and conformal vector fields

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ABSTRACT: The navigation technique is very effective to obtain or classify a Finsler metric from a given a Finsler metric (especially a Riemannian metric) under an action of a vector field on a differential manifold. In this survey, we will survey some recent progress on the navigation problem and conformal vector fields on Finsler manifolds, and their applications in the classifications of some Finsler metrics of scalar (resp. constant) flag curvature.

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1. Introduction

The navigation problem was initially considered by E. Zermelo on the sea plane (i.e., Euclidean plane), and it is to search for the paths of shortest travel time problem for a vessel from one point to another point on the sea plane with a constant speed under the influence of a wind force or a water current, called *Zermelo's navigation problem* ([36]). This is both an optimization and a geometric problem. Recently, this is generalized by Bao-Shen-Robles in [3] for the case when the sea is a Riemannian manifold (M, h) under the assumption that a wind V is a time-independent weak wind, i.e. $h(x, -V_x) < 1$ and by Z. Shen in general setting for the case when the sea is an arbitrary measure space under the perturbation of the external force V ([20], also see [8]).

Further, some deep studies when the wind or current is of a special form, such as a Killing, homothetic or conformal vector field, can be found in [3], [9]-[18], [23]-[24], [27]-[33] etc.. In particular, we can obtain some new Finsler metrics with some curvature properties from a given Finsler metric with some curvature properties under a perturbation of the conformal (resp. Killing or homothetic) vector field. For example, Bao-Robles-Shen classified Randers metrics of constant flag curvature via the navigation technique with the navigation data (h, V), where V is a homothetic vector field on a Riemannian manifold (M, h) of constant sectional curvature ([3]). This is one of the successful applications of the navigation technique in Finsler geometry. From this view, it is important to study conformal (resp. Killing or homothetic) vector fields on a manifold (M, F). On the other hand, conformal vector fields on a manifold M are closely related with the conformal transformations of M. It is interesting to determine

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the dimension of the vector space consisting of conformal (resp. homothetic or Killing) vector fields and hence the structure of the conformal (resp. homothetic or isometric) transformation group on a Finsler manifold (M, F).

It is known that every simply connected and complete Riemannian manifold with constant sectional curvature is isometric to one of three models: sphere $\mathbb{S}^n$, Euclidean space $\mathbb{R}^n$ and hyperbolic space $\mathbb{H}^n$. Note that the flag curvature in Finsler geometry is a natural generalization of sectional curvature in Riemannian geometry. It is a natural problem to classify Finsler metrics of constant flag curvature. This problem is very complicated and is far from settled. In this survey, we first review some progress of the navigation problem in general setting and the conformal vector fields on a Finsler manifold (M, F), and then introduce their applications in studying some Finsler metrics F of the scalar (resp. constant) flag curvature by navigation approach.

2. Navigation problem

Consider an object which is pushed by an internal force and an external force to determine a curve from one point to another in the metric space, along which it takes the least time for the object to travel. This is the navigation problem in general setting. See [20], [8], [22] for more details.

Let (M, F) be a Finsler space. Suppose that an object on (M, F) is pushed by an internal force with the velocity vector U_x of constant length, i.e. $F(x, U_x) = 1$ for any $x \in M$. For any oriented curve $c : [a, b] \to M$, $c = c(t)$ with $\dot{c} = U_{c(t)}$, the time it takes for the object to travel along c is

$$b - a = \int_a^b 1dt = \int_a^b F(c(t), U_{c(t)})dt = Length(c).$$

It follows that without external force acting on the object, any path of shortest time is just the shortest path of F.

Now given an external force W pushing the object, we assume that the force W is weak, i.e., $F(x, -W_x) < 1$, so that the object can move forward in any direction determined by $T_x = U_x + W_x$. Obviously, $T \neq 0$. Let y be a unit vector of T with respect to F, i.e., $T_x = F(x, T_x)y$. Thus

$$F(x, F(x, T_x)y - W_x) = F(x, T_x - W_x) = F(x, U_x) = 1. \tag{2.1}$$

Consider the following equation on $\nu > 0$:

$$F(x, \nu y - W_x) = 1.$$

Since $F(x, -W_x) < 1$, the above equation has a unique solution ν. From this and (2.1), we have $\nu = F(x, T_x) > 0$. Define $\tilde{F} : TM \to [0, \infty)$ by

$$\tilde{F}(x, y) := \begin{cases} \frac{1}{F(x,T_x)}, & \text{if } y \in I_x, \\ F(x, y)\tilde{F}\left(x, \frac{y}{F(x,y)}\right), & \text{others,} \end{cases}$$

where $I_x = \{y \in T_xM | F(x, y) = 1\}$ is the indicatrix at $x \in M$. Plugging the above equality into (2.1) yields

$$F\left(x, \frac{y}{\tilde{F}(x, y)} - W\right) = 1, \quad y \in I_x.$$

It is to check that $\tilde{F}(x, \lambda y) = \lambda\tilde{F}(x, y)$. Thus $\tilde{F}$ is a Finsler metric on M. Further,

$$\tilde{F}(x, T_x) = \tilde{F}(x, F(x, T_x)y) = F(x, T_x)\tilde{F}(x, y) = 1,$$

which shows that the shortest time paths on a Finsler manifold (M, F) under the influence of a weak wind W with $F(x, -W_x) < 1$ are just the geodesics of the new Finsler metric $\tilde{F} = \tilde{F}(x, y)$ defined by the following equation:

$$F\left(x, \frac{y}{\tilde{F}} - W_x\right) = 1, \quad y \in T_xM. \tag{2.2}$$

In particular, if $F = h$ is Riemannian and W is a vector field on M with $h(x, -W_x) < 1$, the resulting metric $\tilde{F}$ in (2.2) is a Randers metric $F = \alpha + \beta$ given by

$$\alpha = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda}, \qquad \beta = -\frac{W_0}{\lambda}, \tag{2.3}$$

where $W_0 := W_i(x)y^i$ with $W_i := h_{ij}W^j$ and $\lambda := 1 - \|W_x\|_h^2 > 0$ with $\|W_x\|_h = \|\beta_x\|_\alpha$. Conversely, every Randers metric $F = \alpha + \beta$ on a manifold M can be constructed from a Riemannian metric h and a vector field W on M. In fact, let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. Define

$$h_{ij} := (1 - b^2)(a_{ij} - b_ib_j), \qquad W^i = -\frac{b^i}{1 - b^2}, \tag{2.4}$$

where $b^i = a^{ij}b_j$ and $b = \|\beta\|_\alpha$. Then F is given by (2.3) for $h = \sqrt{h_{ij}(x)y^iy^j}$ and $W = W^i(x)\frac{\partial}{\partial x^i}$. Moreover, $b = h(x, -W_x) < 1$. Thus there is a one-to-one correspondence between a Randers metric $F = \alpha + \beta$ and a pair (h, W) with $h(x, -W_x) < 1$ on a Riemannian manifold (M, h). We often call (h, W) the *navigation data* or *navigation representation* for F.

When the external force W becomes stronger, i.e., $h(x, -W_x) = 1$, the solution of (2.2) is a Kropina metric $F = \frac{h^2}{2W_0}$, which is a Finsler metric with sigularity, called a *conic Kropina metric*. It is regular on the domain $C = \cup_{x\in M} C_x$, where $C_x = \{y = y^i\frac{\partial}{\partial x^i} \in T_xM | \beta > 0\}$ is a conic domain of T_xM. Conversely, given a Kropina metric $F = \frac{\alpha^2}{\beta}$, define

$$h = \frac{2}{b}\alpha, \qquad W_0 = \frac{2}{b^2}\beta. \tag{2.5}$$

Also, there is a one-to-one correspondence between a Kropina metric $F = \frac{\alpha^2}{\beta}$ and a pair (h, W) with $h(x, -W_x) = 1$. In this case, we also call (h, W) the *navigation data* or *navigation representation* of F. This case can be regarded as a limit of the navigation problem for Randers metrics. It is worth mentioning that the solution space of the navigation problem on a Randers manifold or a Kropina manifold is closed. However, this is not true in general for other cases.

3. Conformal vector fields

Let (M, F) be an n-dimensional Finsler manifold and $(x, y) = (x^i, y^i)$ the local coordinates on the tangent bundle TM. Let φ be a diffeomorphism on M and $\varphi_* : T_xM \to T_{\varphi(x)}M$ be the tangent map at a point x. φ is called a *conformal transformation* on M if there exists a smooth scalar function $c = c(x)$ on M such that

$$F(\varphi(x), \varphi_*(y)) = e^{2c(x)}F(x, y),$$

where $y \in T_xM$ and c is called a *conformal factor* of φ ([1]). A vector field V on (M, F) is called a *conformal vector field* with a conformal factor $c = c(x)$ if the one-parameter transformation group $\{\varphi_t\}$ generated by V is a conformal transformation on (M, F), that is,

$$F(\varphi_t(x), (\varphi_t)_*(y)) = e^{2c_t(x)}F(x, y), \quad \forall x \in M,\ y \in T_xM, \tag{3.1}$$

where $c_t(x) = \int_0^t c(\varphi_s(x))ds$. It is obvious that $c(x) = \frac{dc_t(x)}{dt}|_{t=0}$ and $c_0(x) = 0$ ([12]-[13]). Let Φ_t be a lift on TM of φ_t on M, i.e., $\Phi_t(x, y) := (\varphi_t(x), (\varphi_t)_*(y))$. Then $\{\Phi_t\}$ is a local one-parameter transformation group on TM. It induces a vector field on TM, denoted by X_V. In local coordinates, $X_V = V^i\frac{\partial}{\partial x^i} + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i}$. Thus (3.1) is equivalent to $\Phi_t^*F = e^{2c_t(x)}F$, equivalently,

$$X_V(F) = 2cF.$$

Actually, we have the following equivalent characterizations on conformal vector fields of F.

Proposition 3.1. *([23]) . Let (M, F) be an n-dimensional smooth Finsler manifold and V be a vector field on M. Then, in local coordinates, the following conditions are equivalent.*

(1) $V = V^i\frac{\partial}{\partial x^i}$ is a conformal vector field with the conformal factor $c(x)$;

(2) $\frac{\partial g_{ij}}{\partial x^p}V^p + g_{pj}\frac{\partial V^p}{\partial x^i} + g_{ip}\frac{\partial V^p}{\partial x^j} + 2C_{ijp}\frac{\partial V^p}{\partial x^q}y^q = 4cg_{ij}$;

(3) $V_{i;j} + V_{j;i} + 2C^p_{ij}V_{p;q}y^q = 4cg_{ij}$;

(4) $\frac{\partial F}{\partial x^i}V^i + \frac{\partial F}{\partial y^i}\frac{\partial V^i}{\partial x^j}y^j = 2cF$, namely, $X_V(F) = 2cF$,

where $g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial x^i\partial x^j}$ and $C_{ijp} := \frac{1}{2}\frac{\partial g_{ij}}{\partial x^p}$ are the coefficients of the fundamental tensor g and Cartan torsion C respectively, $(g^{ij}) = (g_{ij})^{-1}$, $C^p_{ij} = g^{pq}C_{ijq}$, $V_i = g_{ij}V^j$ and ";" is the horizontal covariant derivative with respect to the Chern connection of F.

In particular, V is called a *homothetic vector field* with dilation c if c is constant and a *Killing vector field* if $c = 0$ ([17],[23]). If $F = h = h_{ij}(x)y^iy^j$ is Riemannian, then the Cartan tensor vanishes and the corresponding V is exactly a conformal (resp. homothetic or Killing) vector field on a Riemannian manifold (M, h), that is, $V = V^i\frac{\partial}{\partial x^i}$ satisfies

$$V_{i|j} + V_{j|i} = 4ch_{ij}, \tag{3.2}$$

where "$_|$" means the covariant derivative with respect to the Levi-Civita connection of h and $c = c(x)$ is a scalar function (resp. constant, or zero) on M.

When h has constant sectional curvature μ, we can express h in the following projective form:

$$h = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1+\mu|x|^2}, \qquad y \in T_x\mathbb{R}^n. \tag{3.3}$$

Then the general solutions of (3.2) are given by

$$\begin{aligned} V &= 2\left\{\left(\delta\sqrt{1+\mu|x|^2} + \langle a, x\rangle\right)x - \frac{|x|^2 a}{1+\sqrt{1+\mu|x|^2}}\right\} \\ &\quad +Qx + \mu\langle d, x\rangle x + d, \end{aligned} \tag{3.4}$$

where δ and μ are constant, $Q = (q^i{}_j)$ is a skew symmetric matrix in $\mathbb{R}^n$ and $a, d \in R^n$ are constant vectors ([25], also see [4], [22]). In this case, $c = \frac{\delta+\langle a,x\rangle}{\sqrt{1+\mu|x|^2}}$.

For a general Finsler metrics F, which is non-Riemannian, the study of the conformal vector fields is very complicated. In this article, we shall give some equivalent characterizations of conformal vector fields for some special class of Finsler metrics and consequently determine all conformal vector fields on such a class of Finsler manifolds under some conditions based on Proposition 3.1.

3.1. Randers metrics

Randers metrics are the simplest non-Riemannian Finsler metrics, which are expressed by $F = \alpha + \beta$ with $b := \|\beta\|_\alpha < 1$, where

$$\alpha = \sqrt{a_{ij}(x)y^iy^j}, \quad \beta = b_i(x)y^i$$

are respectively a Riemannian metric and 1-form on M.

From (4) in Proposition 3.1, it is easy to see that the following statement holds by the irrationality of α.

Proposition 3.2. *Let V be a vector field on a Randers manifold $(M, F = \alpha + \beta)$. Then V is a conformal vector field with the conformal factor $c = c(x)$ if and only if both $X_V(\alpha) = 2c\alpha$ and $X_V(\beta) = 2c\beta$.*

Observe that

$$X_V(\alpha^2) = 2V_{i;j}y^iy^j, \qquad X_V(\beta) = (V^kb_{j;k} + b^kV_{k;j})y^j. \tag{3.5}$$

Thus, Proposition 3.2 can be rewritten as the following form.

Proposition 3.2$'$. *Let V be a vector field on a Randers manifold $(M, F = \alpha + \beta)$. Then V is a conformal vector field with the conformal factor $c = c(x)$ if and only if $V = V^i\frac{\partial}{\partial x^i}$ satisfies the following system of PDEs*

$$\begin{cases} V_{i;j} + V_{j;i} = 4ca_{ij}, \\ V^jb_{i;j} + b^jV_{j;i} = 2cb_i, \end{cases} \tag{3.6}$$

here we use the Riemannian metric tensor a_{ij} to raise and lower the indices of V or b and ";" is the covariant derivative with respect to the Levi-Civita connection of α.

Remark 3.3. If $c = 0$ in Proposition 3.2 or 3.2$'$, then V is a Killing field on M and the corresponding Killing field equations is exactly $\mathcal{L}_V\alpha = 0$ and $\mathcal{L}_V\beta = 0$, where $\mathcal{L}_V\alpha$ and $\mathcal{L}_V\beta$ are the Lie derivatives of Riemannian metric α and one-form β on M respectively.

From Proposition 3.2, one obtains another equivalent description of conformal vector fields V on a Randers manifold (M, F) in terms of the navigation data (h, W).

Proposition 3.4. *([23]) Let $F = \alpha + \beta$ be a Randers metric on a manifold M and (h, W) be its navigation data given by (2.3) or (2.4). Then a vector field V on (M, F) is a conformal vector field with the conformal factor $c = c(x)$ if and only if $V = V^i \frac{\partial}{\partial x^i}$ satisfies*

$$\begin{cases} V_{i|j} + V_{j|i} = 4ch_{ij}, \\ V^i W_{j|i} + W^i V_{i|j} = 2cW_j, \end{cases}$$

where we use h_{ij} to raise and lower the indices of V and W and "|" is the covariant derivative with respect to the Levi-Civita connection of Riemannian metric h.

We say a Finsler metric F is of *scalar flag curvature* if its flag curvature $K_F = K(x, y)$ independent of the flag including $y \in T_xM$ and it is of *weakly isotropic flag curvature* if $K_F = \frac{3\theta}{F} + \zeta$, where $\theta = \theta_i y^i$ is a 1-form and $\zeta = \zeta(x)$ is a scalar function on M. In particular, if $\theta = 0$ (or $\theta = 0$ and ζ=constant), then we say F is of *isotropic(or constant) flag curvature*. Schur Lemma tells us that F is of constant flag curvature if F is of isotropic flag curvature when dim $(M) \geq 3$ ([8], [22]).

Let $F = \alpha + \beta$ be a Rander metric expressed in terms of the navigation data (h, W) by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}. \tag{3.7}$$

Suppose that F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \zeta$. According to [4] or [26], h is of isotropic sectional curvature $\mu(x)$ (μ =constant when $n \geq 3$) and W is conformal with respect to h, i.e.,

$$W_{i|j} + W_{j|i} = -4\sigma h_{ij}$$

for some scalar function $\sigma = \sigma(x)$ on M. In this case, $\theta = \sigma_{x^i} y^i$ and $\zeta = \mu - \sigma^2 - 2\sigma_{x^i} W^i$. From Proposition 3.4 and (3.3)-(3.4), we can completely determine conformal vector fields on a Randers manifold (M, F) of weakly isotropic flag curvature when $\dim(M) \geq 3$.

Theorem 3.5. *([23]) Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension $n \geq 3$. Suppose V is a conformal vector field with the conformal factor $c(x)$ and F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \zeta$. Then in the same local coordinates for the local standard expression of F, V is given by one of the following*

(1) $V = Qx$. In this case, $c = 0$, $\theta = \frac{\langle v,y \rangle}{\xi} - \frac{\mu(\delta + \langle v,x \rangle)\langle x,y \rangle}{\xi^3}$ and

$$\zeta = \mu - \frac{1}{\xi^2} \left\{ 4\xi(\xi - 1)\delta^2 + [(4\xi - 3)\delta + \langle v, x \rangle] [\delta - 3\langle v, x \rangle] \right\} - \frac{4|x|^2|v|^2}{\xi(1 + \xi)}, \tag{3.8}$$

where $\xi = \sqrt{1 + \mu|x|^2}$, μ and δ are constants, Q is a skew-symmetric constant matrix and v is a nonzero constant vector in $\mathbb{R}^n$ with $Qv = 0$.

(2)

$$V = 2(\epsilon\xi + \langle a, x \rangle) x - \frac{2|x|^2 a}{1 + \xi},$$

where $\xi = \sqrt{1 + \mu|x|^2}$, ϵ and μ are constants, and a is a nonzero constant vector in $\mathbb{R}^n$. In this case, $c = \frac{\epsilon + \langle a,x \rangle}{\xi}$, $\theta = 0$ and $K_F = \zeta = \mu$=constant.

(3) $V = 2\epsilon\xi x + Qx + d + \mu\langle x, d \rangle x$. In this case, $c = \frac{\epsilon}{\xi}$, $\theta = \frac{-\mu\delta\langle x,y \rangle}{\xi^3}$ and $\zeta = \mu - 4\delta^2 + \frac{3\delta^2}{\xi^2}$, where $\xi = \sqrt{1 + \mu|x|^2}$, μ, ϵ and δ are constants with $\delta\epsilon = 0$, Q is a skew-symmetric constant matrix, and d is a constant vector in $\mathbb{R}^n$ with $\delta d = \mu\epsilon d = \epsilon Qx = 0$.

In particular, if the Randers metric F is of constant flag curvature, then we obtain explicit expressions of conformal vector fields on M without restriction on dimension of M as follows.

Theorem 3.6. *([23] Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M of constant flag curvature K_F. Suppose V is a conformal vector field with the conformal factor $c(x)$. Then V is given by one of the following*

(1) V is given by (2) in Theorem 3.5;

(2) $V = 2\epsilon\xi x + Qx + d + \mu\langle x, d \rangle x$, where $\mu \neq 0$ and $\epsilon d = \epsilon Qx = 0$. In this case, $K_F = \mu \neq 0$;

(3) $V = 2\epsilon\xi x + Qx + d$, where $\epsilon Qx = 0$ and $K_F = -\delta^2$, here δ is a constant with $\delta\epsilon = 0$ and $\delta d = 0$.

Remark 3.7. *In Theorems 3.5-3.6, $W = 0$ in the navigation data (h, W) of F if $c(x)$ is not constant. In this case, the Randers metric F is Riemannian. This can be seen from the proof of Theorem 1.2 in [23]. In other words, the conformal vector fields on a Randers manifold (M, F) must be homothetic if F is not Riemannian. This was also proved in [12] in a different way.*

3.2. (α,β)-metrics

As a generalization of Randers metrics, (α,β)-metrics are an important class of Finsler metrics, which are expressed by $F=\alpha\phi(s)$, where $s=\beta/\alpha$ and ϕ is a smooth positive function of one variable s satisfying

$$\phi - s\phi' + (b^2-s^2)\phi'' > 0$$

such that F is a regular Finsler metric. In particular, if $\phi=1+s$, then $F=\alpha+\beta$ is a Randers metric and if $\phi=(1+s)^2$, then $F=\frac{(\alpha+\beta)^2}{\alpha}$ is a square metrics etc.. When $\phi=\frac{1}{s}$, $F=\frac{\alpha^2}{\beta}$ is a Kropina metric, which is a Finsler metric with singularity. There are a large number of references to study (α,β)-metrics (see [8], [22] and references therein). In this subsection, we characterize the conformal vector fields of (α,β)-metrics.

It follows from Proposition 3.1 that

Proposition 3.8. *([15], [22]) Let $F=\alpha\phi(s), s=\frac{\beta}{\alpha}$ be an (α,β)-metric on a smooth manifold M. Suppose $\phi'(0)\neq 0$. Then V is a conformal vector field with the conformal factor $c=c(x)$ on (M,F) if and only if α and β satisfy $X_V(\alpha)=2c\alpha$ and $X_V(\beta)=2c\beta$.*

Locally, by (3.5), $X_V(\alpha)=2c\alpha$ and $X_V(\beta)=2c\beta$ are equivalent to (3.6). It is not known that if there exists the navigation version for (α,β)-metrics up to now. It is a question how to determine all conformal vector fields on (α,β)-manifolds.

Since the conic Kropina metric $F=\frac{\alpha^2}{\beta}$ is an (α,β)-metrics with singularity, Proposition 3.8 can not be directly applied to F. For this metric F, Cheng-Yin-Li gave an equivalent characterization of conformal vector fields on (M,F) by Proposition 3.1.

Proposition 3.9. *([9]) Let $F=\frac{\alpha^2}{\beta}$ be a Kropina metric on a smooth manifold M. Then a vector field V on (M,F) is a conformal vector field with conformal factor $c=c(x)$ if and only if $V=V^i\frac{\partial}{\partial x^i}$ satisfies*

$$\begin{cases} V_{i;j}+V_{j;i}=4\tau a_{ij}, \\ V^i b_{j;i}+b^i V_{i;j}=2(2\tau-c)b_j, \end{cases}$$

where $\tau=\tau(x)$ is a scalar function on M, we use a_{ij} to raise and lower the indices of V^i and b_i and ";" is the covariant derivative with respect to the Levi-Civita connection of α.

In terms of the navigation data (h,W), we write the Kropina metric $F=\frac{h^2}{2W_0}$. Then we have another equivalent characterization for conformal vector fields based on Proposition 3.8 and (2.5).

Proposition 3.10. *([5]) Let $F=\frac{\alpha^2}{\beta}$ be a Kropina metric with the navigation data (h,W). Then a vector field V on (M,F) is a conformal vector field with conformal factor $c=c(x)$ if and only if $V=V^i\frac{\partial}{\partial x^i}$ satisfies*

$$\begin{cases} V_{i|j}+V_{j|i}=4ch_{ij}, \\ V^i W_{j|i}+W^i V_{i|j}=2cW_j, \end{cases}$$

where $\tau=\tau(x)$ is a scalar function on M, we use h_{ij} to raise and lower the indices of V^i and b_i and "|" is the covariant derivative with respect to the Levi-Civita connection of h.

Similar to the proof of Theorem 3.5, all conformal vector fields on $(M,F=\alpha^2/\beta)$ with weakly isotropic flag curvature are given by the following form.

Theorem 3.11. *([5], [9]) Let $F=\frac{\alpha^2}{\beta}$ be a Kropina metric on a manifold M of dimension $n\geq 3$. Suppose that V is a conformal vector field on M with conformal factor $c(x)$ and F is of weakly isotropic flag curvature $K_F=\frac{3\theta}{F}+\zeta$. Then in the same local coordinate system for the local standard expression of F, V is given by one of the following*

(1) $V=d+2\epsilon x$, where ϵ is a constant and d is a nonzero constant vector in $\mathbb{R}^n$ with $|d|=1$. In this case, $c=\epsilon$, $\theta=0$ and $K_F=0$.

(2) $V=Qx+d+\mu\langle d,x\rangle x$, where μ is a positive constant, d is a nonzero constant vector in $\mathbb{R}^n$ with $|d|=1$, and Q is a skew symmetric matrix with $Qd=0$ and $Q^TQ+\mu dd^T=\mu I$, where I is an identity matrix. In this case, $c=0$, $\theta=0$ and $K_F=\zeta=\mu$.

In both cases, V is a homothetic vector field on M.

3.3. General (α, β)-metrics

General (α, β)-metrics form a more abroad class of Finsler metrics, which can be expressed in the form $F = \alpha\phi(b^2, s)$, where ϕ is a smooth positive function of two variables $b^2 := \|\beta\|_\alpha^2$ and $s := \beta/\alpha$ satisfying

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \qquad |s| \le b < b_0$$

when $n \ge 3$, which garantee that F is a regular Finsler metric, here ϕ_2 and ϕ_{22} are the first and second partial derivatives of ϕ with respect to the second variable s respectively ([35]). Similarly, we shall denote by $\phi_1, \phi_{11}, \phi_{12}$ the first, second and the mixed derivatives of ϕ with respect to the variables t and s. Obviously, if ϕ is independent of b^2, then the general (α, β)-metric is exactly the (α, β)-metric. However, many well known (α, β)-metrics are general (α, β)-metrics, for example, Randers metrics and square metrics are defined by functions $\phi = \phi(b^2, s)$ in the following form

$$\phi = \frac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2}, \qquad \phi = \frac{(\sqrt{1 - b^2 + s^2} + s)^2}{(1 - b^2)^2\sqrt{1 - b^2 + s^2}}.$$

Besides, spherically symmetric metrics $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ are a special class of general (α, β)-metrics, where $\langle , \rangle$ is a standard inner product in $\mathbb{R}^n$ and $|\cdot|$ is a norm with respect to $\langle , \rangle$. These metrics are widely studied ([8], [22], [27], [30], [35] and references therein).

It is not known if there exists the navigation version for general (α, β)-metrics up to now. However, based on Proposition 3.1, we have

Proposition 3.12. *([30]) Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a regular general (α, β)-metric on an $n(\ge 3)$-dimensional manifold M. Assume that $\phi_2(b^2, 0) \ne 0$. Then a vector field V on (M, F) is a conformal vector field with conformal factor $c(x)$ if and only if there exists a local coordinate system, in which there are scalar functions $\sigma = \sigma(x)$ and $\tau := \frac{\phi_1(b^2,0)}{\phi(b^2,0)}$ on M such that V satisfies the following system of PDEs*

$$V_{i;j} + V_{j;i} = 2\left[2c - \tau X_V(b^2)\right] a_{ij} + \frac{2}{b^2}\left(\sigma + \tau\right) X_V(b^2) b_i b_j, \tag{3.9}$$

$$V^j b_{i;j} + b^j V_{j;i} = \left[2c + \left(\sigma + \frac{1}{2b^2}\right) X_V(b^2)\right] b_i, \tag{3.10}$$

and ϕ satisfies

$$X_V(b^2)\left\{2b^2\phi_1 + s\left[1 + 2(\sigma + \tau)(b^2 - s^2)\right]\phi_2 - 2\left[\tau b^2 - (\sigma + \tau)s^2\right]\phi\right\} = 0, \tag{3.11}$$

where we use $\alpha = (a_{ij})$ to raise or lower the indices of V^i and b_i, and ";" denotes the covariant derivative with respect to Levi-Civita connection of α.

For the sake of convenience, set

$$\Psi := 2b^2\phi_1 + s\left[1 + 2(\sigma + \tau)(b^2 - s^2)\right]\phi_2 - 2\left[\tau b^2 - (\sigma + \tau)s^2\right]\phi.$$

If $\phi = \phi(s)$ is an (α, β)-metric, i.e., $\phi_1 = 0$, then $\Psi = s(1 + 2\sigma b^2)\phi_2 + 2\left[(\sigma + \tau)s^2 - \tau b^2\right](\phi - s\phi_2) \ne 0$. In fact, letting $s = 0$ in $\Psi = 0$ yields $\tau = 0$ and $\sigma = -\frac{1}{2b^2}$. Plugging them back in $\Psi = 0$ gives that $\sigma(\phi - s\phi_2) = 0$, which is impossible since F is regular. Thus, $X_V(b^2) = 0$ by (3.11) and hence Proposition 3.12 is reduced to Proposition 3.8. In particular, it is reduced to Proposition 3.2 for Randers metrics. For general (α, β)-metrics $F = \alpha\phi(b^2, s)$, $X_V(b^2) = 0$ does not always hold (see Corollary 3.13 below). When ϕ is an expansion of a Taylor series in s, i.e., $\phi = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + o(s^3)$, where $a_i = a_i(b^2)(i \ge 0)$ with $a_1 = \phi_2(b^2, 0) \ne 0$, we have $a_0' = a_0\tau$. Take $\sigma = -\frac{1}{2b^2} - \frac{a_1'}{a_1}$. It is easy to check that

$$\Psi \quad = \quad -2a_0\left\{\frac{1}{2b^2} + \frac{a_1'}{a_1} - \frac{a_0'}{a_0} + \left[\frac{a_2}{a_0}\left(2\frac{a_1'}{a_1} - \frac{a_0'}{a_0}\right) - \frac{a_2'}{a_0}\right]b^2\right\}s^2 + o(s^2).$$

On the other hand, we have by a direct calculation

$$\begin{aligned} &(\phi - s\phi_2)b^{-2}\left[(b^2 - s^2)\left(\frac{a_1'}{a_1} - \frac{a_0'}{a_0}\right) - \frac{s^2}{2b^2}\right] + \phi_1 - \phi\frac{a_1'}{a_1} \\ &= -\frac{a_0}{b^2}\left\{\frac{1}{2b^2} + \frac{a_1'}{a_1} - \frac{a_0'}{a_0} + \left[\frac{a_2}{a_0}\left(2\frac{a_1'}{a_1} - \frac{a_0'}{a_0}\right) - \frac{a_2'}{a_0}\right]b^2\right\}s^2 + o(s^2). \end{aligned}$$

Thus, Proposition 3.12 is reduced to Theorems 1.1 and 1.2 in [27] in this case. It is worth mentioning that Chen-Mo recently studied the conformal vector fields for general (α, β)-metrics $F = \alpha\phi(b^2, s)$ with $\phi_2(b^2, 0) = 0$ ([6]).

Assume that there is a function $\tau = \tau(b^2)$ such that ϕ satisfies that

$$\phi_1 = \tau(\phi - s\phi_2). \tag{3.12}$$

Then $\tau = \frac{\phi_1(b^2,0)}{\phi(b^2,0)}$ as in Proposition 3.12. It is easy to see that the general solution of (3.12) is given by

$$\phi(b^2, s) = sf\left(\frac{1}{s}\exp\left(\int \tau(b^2)db^2\right)\right), \tag{3.13}$$

where $f = f(t)$ is a smooth function of one variable t. From (3.12), we get

$$\Psi = s\left\{(1 + 2\sigma b^2)\phi_2 + 2(\sigma + \tau)s(\phi - s\phi_2)\right\}.$$

Therefore, $\Psi = 0$ if and only if $\tau = -\sigma = \frac{1}{2b^2}$ since $\phi_2(b^2, 0) \neq 0$. In particular, when $\tau \neq -\frac{1}{2b^2}$ or $\sigma + \tau \neq 0$, we have $\Psi \neq 0$ and hence $X_V(b^2) = 0$ by Proposition 3.12 if V is a conformal vector field. Thus we have

Corollary 3.13. *([30]) Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a general (α, β)-metric on an $n(\geq 3)$-dimensional manifold M. Assume that $\phi_2(b^2, 0) \neq 0$ and there is a function $\tau = \tau(b^2)$ such that $\phi_1 = \tau(\phi - s\phi_2)$. Then a vector field V on (M, F) is a conformal vector field with conformal factor $c(x)$ if and only if there exists a local coordinate system, in which there is a scalar function $\sigma = \sigma(x)$ on M such that V satisfies (3.9)-(3.10) and ϕ satisfies*

$$X_V(b^2)\left\{(1 + 2\sigma b^2)\phi_2 + 2(\sigma + \tau)s(\phi - s\phi_2)\right\} = 0.$$

Further, if $\tau = -\sigma = \frac{1}{2b^2}$, then V is a conformal vector field on (M, F) if and only if V satisfies

$$V_{i;j} + V_{j;i} = 2\left(2c - \frac{1}{2b^2}X_V(b^2)\right)a_{ij},$$
$$V^j b_{i;j} + b^j V_{j;i} = 2cb_i.$$

In this case, $\phi = sf(\frac{b}{s})$, where f is a smooth function of one variable.

If $\tau \neq \frac{1}{2b^2}$ or $\sigma + \tau \neq 0$, then V is a conformal vector field on (M, F) if and only if V satisfies $X_V(b^2) = 0$ and

$$V_{i;j} + V_{j;i} = 4ca_{ij},$$
$$V^j b_{i;j} + b^j V_{j;i} = 2cb_i.$$

In this case, ϕ is given by (3.13).

Observe that ϕ expressed by (3.13) has singular points unless $f(t) = t$. However, if $f(t) = t$, then F is Riemannian, which is excluded since $\phi_2(b^2, 0) \neq 0$. Thus, the corresponding F is a Finsler metric with singularity. In particular, let $f(t) = t^2$. Then $F = \nu(b^2)\frac{\alpha^2}{\beta}$, which is actually a Kropina metric, where $\nu(b^2) := f(\exp(2\int \tau(b^2)db^2))$. In this case, Corollary 3.13 is exactly proposition 3.9.

Since the system of PDEs (3.9)-(3.10) and (3.11) are highly nonlinear, it is difficult to solve (3.9)-(3.11). However, under some restrictions on F, we can determine all solutions V and ϕ of (3.9)-(3.11). For this, let α has constant curvature μ and β is a conformal 1-form with respect to α, i.e., there exists a scalar function $\rho = \rho(x)$ on M such that

$$b_{i;j} + b_{j;i} = 4\rho a_{ij}.$$

In the same way as (3.3)-(3.4), α is locally isometric to the following projective form

$$\alpha = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu < x, y >^2}}{1 + \mu|x|^2}, \qquad y \in T_x\mathbb{R}^n, \tag{3.14}$$

and $\beta = (b_i(x))$ is given by

$$b_i = a_{ij}b^j = 2\xi^{-3}(\delta + \langle a, x\rangle)x_i + \xi^{-2}q_{ij}x^j + \xi^{-2}d_i - 2\xi^{-2}(1 + \xi)^{-1}|x|^2a_i, \tag{3.15}$$

where $\xi = \sqrt{1 + \mu|x|^2}$, δ is a constant, we use δ_{ij} to raise or lower the indices of the vectors x, d and a in $\mathbb{R}^n$, and $q_{ij} = \delta_{ik}q^k_{\ j}$. In this case, $\rho = \xi^{-1}(\delta + \langle a, x\rangle)$ (also see [8], Lemma 5.2.9 and Proposition 5.2.10).

Proposition 3.14. *([30]) Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a general (α,β)-metric on an $n(\geq 3)$-dimensional manifold M and V be a conformal vector field on (M,F) with conformal factor $c(x)$. Assume that $\phi_2(b^2,0) \neq 0$ and there is a function $\tau = \tau(b^2) \neq \frac{1}{2b^2}$ such that $\phi_1 = \tau(\phi - s\phi_2)$. If α has constant curvature μ and β is a conformal 1-form with respect to α, then in the same local coordinate system for the local expression of F, ϕ is given by (3.13) and V is given by one of the following cases.*

(1) $V = Qx$, where $Q = (q^i{}_j)$ is a skew symmetric constant matrix. In this case, $c = 0$, α and $\beta = (b_i(x))$ are respectively given by (3.14) and (3.15) with $Qv = d = \mu Qx = 0 (v \neq 0)$

(2) $V = 2\epsilon\xi x + Qx + d + \mu\langle d, x\rangle x$, where $\xi = \sqrt{1+\mu|x|^2}$, ϵ is a constant, d is a constant vector in $\mathbb{R}^n$ and Q is a skew symmetric constant matrix with $\epsilon Qx = \mu\epsilon d = \delta d = \delta\mu Qx = 0$. In this case, $c = \epsilon/\xi$, α and $\beta = (b_i(x))$ are respectively given by (3.14) and (3.15) with $v = \epsilon Qx = \mu\epsilon d = \delta d = \delta\mu Qx = 0$ and $\epsilon\delta = 0$.

(3) $V = 2(\epsilon\xi + \langle a, x\rangle)x - 2(1+\xi)^{-1}|x|^2 a$, where ξ, ϵ are the same as (2) and a is a nonzero constant vector in $\mathbb{R}^n$. In this case, $c = \xi^{-1}(\epsilon + \langle a, x\rangle)$, α and $\beta = (b_i(x))$ are respectively given by (3.14) and (3.15) with $v = Qx = d = 0$ and $\delta = 0$.

For spherically symmetric metrics $F = |y|\phi(|x|^2, \frac{\langle x,y\rangle}{|y|})$, $\alpha = |y|$ is an Euclidean metric and $\beta = \langle x, y\rangle$ is a conformal 1-form on $\mathbb{R}^n$. In this case, $\mu = c = 0$. From Proposition 3.14, one obtains

Corollary 3.15. *([30]) Let $F = |y|\phi(|x|^2, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric metric on $\mathbb{R}^n$ and V be a conformal vector field on $(\mathbb{R}^n, F)$ with conformal factor $c(x)$. Assume that $\phi_2(b^2,0) \neq 0$ and there is a function $\tau = \tau(b^2) \neq \frac{1}{2b^2}$ such that $\phi_1 = \tau(\phi - s\phi_2)$. Then ϕ is given by (3.13) and V is given by $V = Qx$, where Q is a skew symmetric constant matrix. In this case, $c = 0$.*

4. Applications

In this section, we introduce some applications of the navigation approach in Finsler spaces of scalar (resp. constant) flag curvature. In particular, classifications of some special Finsler spaces with constant flag curvature are given.

4.1. Randers metrics of scalar flag curvature

Let (M, F, m) be a Finsler measure space. Take a local coordinates $\{x^i\}_{i=1}^n$ around $x \in M$ such that $dm = \sigma(x)dx$. For any $y \in T_xM \setminus \{0\}$, define

$$\tau(x,y) = \ln\left(\frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)}\right).$$

τ is called the *distortion* of (M, F, m).

Let $\gamma(t)$ be a geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = y$. Define

$$S(x,y) := \frac{d}{dt}\left[\tau(\dot\gamma(t))\right]|_{t=0}.$$

We call $S(x,y)$ the *S-curvature* on (M,F) with respect to the measure m. S-curvature was first appeared in [19]. It measures the rate of changes of the distortion τ along the geodesic γ. A Finsler metric F is said to be of *isotropic S-curvature* $c(x)$ if $S(x,y) = (n+1)cF$ for some scalar function $c(x)$. If $c(x)$=constant, then we say F is of *constant S-curvature* c. For a Randers metric $F = \alpha + \beta$ with the navigation data (h, W), F is of isotropic S-curvature $c(x)$ if and only if W satisfies

$$W_{i|j} + W_{j|i} = -2ch^2, \tag{4.1}$$

where " $_|$" means the covariant derivative with respect to h, i.e., W is a conformal vector field of h ([31]).

Proposition 4.1. *([3]) For a Randers metric $F = \alpha + \beta$ with the navigation data (h, W), it has constant flag curvature $K_F = k$ if and only if h has constant sectional curvature $\bar K = k + c^2$ and W satisfies (4.1) in which c is a constant.*

When h is of constant sectional curvature, it is easy to solve (4.1) for W to obtain a complete list of local structure of Randers metrics of constant flag curvature from Proposition 4.1.

Theorem 4.2. *([3]) Let $F = \alpha + \beta$ be a Randers metric on a manifold M with the navigation data (h, W). Then F has constant flag curvature if and only if at any point, there is a local coordinate system in which h is given by (3.3) and W is given by*

$$W = -2c\sqrt{1+\mu|x|^2}x + Qx + \mu\langle d, x\rangle x + d,$$

where c and μ are constants with $c\mu = 0$, $Q = (q^i{}_j)$ is a skew symmetric matrix and $d \in \mathbb{R}^n$ is a constant vector. In this case, the flag curvature is given by $K_F = \mu - c^2$.

Recall that

Theorem 4.3. *([7]) Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature $K_F(x, y)$. Suppose that the S-curvature is isotropic, i.e., $S = (n+1)c(x)F(x, y)$, where $c(x)$ is a scalar function on M. Then there is a scalar function $\zeta(x)$ on M such that*

$$K_F = \frac{3c_{x^i}y^i}{F} + \zeta. \tag{4.2}$$

In particular, $c(x) = c$ is a constant if and only if $K_F = K(x)$ is a scalar function on M.

A natural question arises: does (4.2) imply that the S-curvature is isotropic? The answer is affirmative for Randers metrics. In fact, any Randers metric of weakly isotropic flag curvature is of isotropic S-curvature (Theorem 1.2, [26]). Thus the condition " scalar flag curvature and isotropic S-curvature" and "weakly isotropic flag curvature" are equivalent for Randers metrics.

Using the navigation data (h, W), we can reexpress the formula of the Riemann curvature tensor $R^i{}_k$ of F (see (3.8) in [4]). From this we can prove

Proposition 4.4. *([4]) Let F be a Randers metric on n-dimensional manifold M with the navigation data (h, W). Suppose that F has isotropic S-curvature $c(x)$. Then F is of scalar flag curvature if and only if h is of sectional curvature μ, where $\mu = \mu(x)$ is a scalar function (=constant if $n \geq 3$). In this case, the flag curvature of F is given by*

$$K_F = \frac{3c_{x^i}y^i}{F} + \zeta, \tag{4.3}$$

where $\zeta := \mu - c^2 - 2c_{x_i}W^i$.

Note that W satisfies (4.1) when the Randers metric F has isotropic S-curvature. If h has constant curvature μ, then one can easily solve (4.1) for W and obtain the list of local structures of Randers metrics of scalar flag curvature and isotropic S-curvature from Proposition 4.4.

Theorem 4.5. *([4], [26]) Let $F = \alpha + \beta$ be a Randers metric with the navigation data (h, W) on a manifold M of dimension $n \geq 3$. Then F is of scalar flag curvature and isotropic S-curvature $c(x)$ (or weakly isotropic flag curvature) if and only if at any point, there is a local coordinate system in which h is expressed by (3.3), $c = \frac{\delta + \langle a, x\rangle}{\sqrt{1+\mu|x|^2}}$ and W is given by*

$$W = -2\left\{\left(\delta\sqrt{1+\mu|x|^2} + \langle a, x\rangle\right)x - \frac{|x|^2 a}{\sqrt{1+\mu|x|^2}+1}\right\} + Qx + \mu\langle d, x\rangle + d,$$

where δ, μ are constants, $Q = (q^i{}_j)$ is a skew symmetric matrix and $a, d \in \mathbb{R}^n$ are constant vectors. In this case, the flag curvature K_F is given by (4.3).

Since every Randers metric of constant flag curvature must have constant S-curvature, the class of Randers metrics with isotropic S-curvature and scalar flag curvature contains all Randers metrics of constant flag curvature.

Example 4.1. *([21]) In Theorem 4.5, let $\mu = \delta = 0$, $Q = 0$ and $d = 0$. Then $h = |y|, c = \langle a, x\rangle$ and $W = -2\langle a, x\rangle x + |x|^2 a$. By (3.7), the Randers metric is given by*

$$\begin{aligned} F &= \frac{\sqrt{(1-|a|^2|x|^4)|y|^2 + (|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle)^2}}{1-|a|^2|x|^4} \\ &\quad -\frac{|x|^2\langle a, y\rangle - 2\langle a, x\rangle\langle x, y\rangle}{1-|a|^2|x|^4}. \end{aligned}$$

The S-curvature and the flag curvature are given by

$$S = (n+1)\langle a, x\rangle F, \qquad K_F = \frac{3\langle a, y\rangle}{F} + 3\langle a, x\rangle^2 - 2|a|^2|x|^2.$$

Recall that a Randers metric $F = \alpha + \beta$ is of Douglas type if and only if β is closed ([8], [22]). For Randers metrics of scalar flag curvature, we have

Theorem 4.6. *([28]) Let $F = \alpha + \beta$ be a Randers metric of Douglas type on an $n(\geq 3)$-dimensional manifold M and V be a conformal vector field on (M, F) with a conformal factor $c(x)$. Let $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric derived from (F, V) by (2.2). Then each two of the followings imply the third one.*

1) F is of scalar flag curvature.

2)$\tilde{F}$ is of scalar flag curvature.

3) V is homothetic or $\beta = 0$.

In this case, the flag curvatures K_F and $K_{\tilde{F}}$ of F and $\tilde{F}$ are related by

$$K_{\tilde{F}}(x, y) = K_F(x, \xi) - c^2 - \frac{3c_0}{\tilde{F}} + 2V(c),$$

where $\xi = y - \tilde{F}V$, $\tilde{F} = \tilde{F}(x, y) = F(x, \xi)$ and $c_0 = \frac{\partial c}{\partial x^i} y^i$.

Theorem 4.6 implies that a Randers metric of scalar flag curvature cannot be generated from a given Douglas Randers metric F of scalar flag curvature and a conformal vector V of F by solving the navigation problem unless F is Riemannian or V is homothetic.

Question. It is open to classify Randers metrics of scalar flag curvature.

4.2. Kropina metrics of scalar flag curvature

In this subsection, we shall discuss the classification of Kropina metrics of scalar flag curvature via the navigation technique. First result is due to R. Yoshikawa and K. Okubo.

Proposition 4.7. *([32], [33]) Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an $n(\geq 2)$-dimensional manifold M with the navigation data (h, W). Then F is of constant flag curvature if and only if h is of (nonnegative) constant sectional curvature and W is a Killing vector field of h.*

For Kropina metrics F of weakly isotropic flag curvature, the present author in this paper gave an equivalent characterization in [29].

Proposition 4.8. *([29]) Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an $n(\geq 2)$-dimensional manifold M and $b = \|\beta\|$ be a constant. Then F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \zeta$ if and only if the sectional curvature of α is of nonnegative scalar function $\mu(x)$ and β is a Killing 1-form with respect to α. In this case, $K_F = \zeta = \frac{1}{4}\mu b^2$ and $\theta = 0$.*

Arguments as Randers metrics, the Riemannian metric α and 1-form β can be completely determined. By (2.5), we obtain the local structure of Kropina metrics of weakly isotropic flag curvature.

Theorem 4.9. *([29]) Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an $n(\geq 3)$-dimensional manifold M with the navigation data (h, W). Then F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \zeta$ if and only if at every point there is a local coordinate system in which h and W are given by one of the following*

(1) $h = |y|$ is an Euclidean metric and W is a nonzero constant vector field in $\mathbb{R}^n$. In this case, $F = \frac{|y|^2}{2W_0}$ is a Minkowski metric with $K_F = 0$.

(2)

$$\begin{aligned} h &= \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1 + \mu|x|^2}, \\ W &= \frac{1}{2}\left(Qx + \mu\langle d, x\rangle x + d\right), \end{aligned}$$

where μ is a positive constant, d is a nonzero constant vector with $|d| = 2$, and Q is a skew symmetric matrix with $Qd = 0$ and $Q^TQ + \mu dd^T = 4\mu I$, where I is an identity matrix. In this case, $\|W\|_h = 1$, $W_0 = \langle W, y\rangle_h = \frac{\langle Qx+d, y\rangle}{2(1+\mu|x|^2)}$ and $F = \frac{h^2}{2W_0}$ is a Kropina metric with $K_F = \kappa = \mu > 0$ and $\theta = 0$.

Note that the following conditions are equivalent for Kropina metrics F: (a) F is of scalar flag curvature and isotropic S-curvature S; (b) F is of weakly isotropic flag curvature; (c) F is of isotropic flag curvature. In this case, $S = 0$ ([29]). Thus, Theorem 4.9 also give the local classification for Kropina metrics of constant flag curvature when $\dim(M) \geq 3$.

Example 4.2. *([29])* Let (M, h) be a 3-dimensional Riemannian manifold with positive constant sectional curvature μ. Locally, h is given (3.3). Let

$$Q = \sqrt{\mu}\begin{pmatrix} 0 & p & q \\ -p & 0 & l \\ -q & -l & 0 \end{pmatrix}, \quad d = \pm\begin{pmatrix} l \\ -q \\ p \end{pmatrix} \neq 0,$$

where p, q, l are constants with $p^2+q^2+l^2 = 4$. It is easy to check that Q and d satisfy $Qd = 0$ and $Q^TQ+\mu dd^T = 4\mu I$. Define

$$W = \frac{1}{2}\left(Qx + \mu\langle d, x\rangle x + d\right).$$

Then, by a direct calculation, W_0 is given by

$$\begin{aligned} W_0 &= \frac{\langle Qx + d, y\rangle}{2(1+\mu|x|^2)} = \pm\frac{ly^1 - qy^2 + py^3}{2(1+\mu|x|^2)} \\ &+ \frac{\sqrt{\mu}\left[(px^2 + qx^3)y^1 - (px^1 - lx^3)y^2 - (qx^1 + lx^2)y^3\right]}{2(1+\mu|x|^2)}, \end{aligned}$$

Then, $F = \frac{h^2}{2W_0}$ is a Kropina metric on M with $K_F = K_h = \mu$.

Question. It is open to classify Kropina metrics of scalar flag curvature.

4.3. Finsler metrics of scalar flag curvature

In previous sections, we can see that the navigation technique plays an important role in classifying Randers metrics or Kropina metrics of constant flag curvature. For the general case, the progress of classifying Finsler metrics of scalar (resp. constant) flag curvature is very limited. However we may establish some relationships of the flag curvature $K_{\tilde{F}}$ of the new Finsler metric $\tilde{F}$ and the flag curvature K_F of a given Finsler metric F by the navigation problem. Based on this, we obtain a series of Finsler metrics with some curvature properties.

Theorem 4.10. *([17]) Let $F = F(x, y)$ be a Finsler metric on a manifold M and V a vector field on M with $F(x, -V_x) < 1$. Let $\tilde{F} = \tilde{F}(x, y)$ denote the Finsler metric on M defined by (2.2). Suppose that V is homothetic with dilation c. Then the flag curvatures of $\tilde{F}$ and F are related by*

$$K_{\tilde{F}}(y, y \wedge u) = K_F(\tilde{y}, \tilde{y} \wedge u) - c^2,$$

where $\tilde{y} = y - F(x, y)V$.

Corollary 4.11. *([17]) Let F be a Finsler metric on a manifold M and V its vector field with $F(x, -V_x) < 1$. Let $\tilde{F} = \tilde{F}(x, y)$ denote the Finsler metric on M derived from (2.2). Suppose that V is a homothetic field with dilation c. If F is of scalar (resp. constant) curvature, then $\tilde{F}$ is also of scalar (resp. constant) flag curvature.*

As a generalization of Theorem 4.10, Huang-Mo further obtain the following result.

Theorem 4.12. *([14]) Let $F = F(x, y)$ be a Finsler metric on a manifold M with the Cartan torsion C and V a vector field on M with $F(x, -V_x) < 1$. Let $\tilde{F} = \tilde{F}(x, y)$ denote the Finsler metric on M derived from (2.2). Suppose that V is a conformal vector fields with conformal factor $c(x)$. Then the flag curvatures of $\tilde{F}$ and F are related by*

$$K_{\tilde{F}}(y, y \wedge u) - \left(\frac{3c_{x^i}y^i}{\tilde{F}} - c^2 + 2V(c)\right) = K_F(\tilde{y}, \tilde{y} \wedge u) - 2\frac{C_{(x,[\tilde{y}])}(u, \nabla c, u)}{h_{(x,[\tilde{y}])}(u, u)},$$

where $\tilde{y} = y - F(x, \tilde{y})V$.

Theorems 4.10 and 4.12 shows that we can construct a series of Finsler metrics of scalar (resp. constant) flag curvature from a given a Finsler metric F of scalar (resp. constant) flag curvature and a conformal (resp. homothetic, Killing) vector field V of F with $F(x, -V_x) < 1$. Some examples can be found in [14] and [17]. Anyway, it is open to classify Finsler metrics of scalar (resp. constant) flag curvature.

Besides the relationships of the flag curvatures in Theorems 4.10 and 4.12, the relationships between Ricci curvature, S-curvature, Landsberg curvature for the new metric $\tilde{F}$ generating from the navigation problem and those of a given Finsler metric F were given in [23], [9] and [11]. The navigation technique is also applied to study geodesics in Finsler spaces ([18], [13]), Einstein Finsler metrics ([2], [37]), and the locally dually flat Finsler metrics ([16], [34]) etc.. Further applications of the navigation problem and the conformal vector fields on Finsler manifolds will be studied.

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