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Original Article

Flag curvatures of the unit sphere in a Minkowski-Randers space

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ABSTRACT: On a real vector space V, a Randers norm $\hat{F}$ is defined by $\hat{F} = \hat{\alpha} + \hat{\beta}$, where $\hat{\alpha}$ is a Euclidean norm and $\hat{\beta}$ is a covector. We show that the unit sphere Σ in the Randers space $(V, \hat{F})$ has positive flag curvature, if and only if $|\hat{\beta}|_{\hat{\alpha}} < (5 - \sqrt{17})/2 \approx 0.43845$, thus answering a problem proposed by Prof. Zhongmin Shen. Moreover, we prove that the flag curvature of Σ has a universal lower bound -4.

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(*Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration*)

1. Introduction

Let $(V, \hat{F})$ be a Minkowski space of dimension $n+1$ and let $\Sigma = \{x \in V \mid \hat{F}(x) = 1\}$ be the unit sphere in V. By definition, Σ is a strongly convex hypersurface in V.

There are two natural geometric structures defined on the punctured space $V \setminus \{0\}$. The first is a Riemannian metric defined by the Hessian of $\frac{1}{2}\hat{F}^2$, called the Hessian metric or fundamental tensor. As a Riemannian submanifold, Σ has extrinsic curvature $+1$ (see [1, §14.5]). The second one is the Finsler metric $\hat{F}$ itself. As a Finsler submanifold, Σ has a induced Finsler metric F. When $\hat{F}$ is a Euclidean norm, it is well known that (Σ, F) has constant sectional curvature $+1$. If $\hat{F}$ is not Euclidean, then one cannot expect (Σ, F) to have constant curvature. It is natural to ask, does (Σ, F) always have positive flag curvature? This question is the 25th in Prof. Zhongmin Shen's list of open problems [6]. In this short note, we will provide a negative answer to this question.

We consider a special class of Minkowski norms, namely, Randers norms. In this case, the induced metric on Σ is also of Randers type. One may consult [3, 2, 7, 8] for related results on Randers manifolds.

Recall that a Randers norm $\hat{F}$ is defined by $\hat{F} = \hat{\alpha} + \hat{\beta}$, where $\hat{\alpha}$ is a Euclidean norm and $\hat{\beta}$ is a linear function on V with $|\hat{\beta}|_{\hat{\alpha}} < 1$. The main result of this paper is the following

Theorem 1.1. *Let $\hat{F} = \hat{\alpha} + \hat{\beta}$ be a Randers norm on an $n+1$ dimensional real vector space V. Let $\epsilon := |\hat{\beta}|_{\hat{\alpha}}$, $\epsilon_0 = (5 - \sqrt{17})/2$ and let Σ be the unit sphere in V. If $\epsilon < \epsilon_0$, then Σ has positive flag curvature everywhere; if*

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$\epsilon = \epsilon_0$, then Σ is non-negatively curved and the zero flag curvature only appears at the equator of Σ; if $\epsilon > \epsilon_0$, then there are flags on the equator of Σ with negative flag curvature.

According to this theorem, the unit sphere in a Minkowski space need not to be positively curved. This strange phenomenon shows that the concept of flag curvature could be more complicated than we usually thought (one may compare the survey article [4]).

When $|\hat{\beta}|_{\hat{\alpha}} > \epsilon_0 \approx 0.43845$, it is now known that Σ may posess negative curvature somewhere. What will happen when $|\hat{\beta}|_{\hat{\alpha}}$ approaches 1?

Theorem 1.2. *Let $\hat{F} = \hat{\alpha} + \hat{\beta}$ be a Randers norm on an $n+1$ dimensional real vector space V and let Σ be the unit sphere in V. Let $\epsilon := |\hat{\beta}|_{\hat{\alpha}}$, then the lower bound of the flag curvature of the induced metric on Σ is $\frac{1}{2}(1+\epsilon)^2(\epsilon^2 - 5\epsilon + 2)$. When $\epsilon \to 1$, the above lower bound approaches its minimal value -4.*

The paper is organized as follows. In Section 2, we will review some basic facts on Randers manifolds and derive a formula for the induced metric on Σ. In Section 3, we compute the Riemann curvature tensor of the induced metric and analyze its eigenvalues. In Section 4, we will investigate the sign of the flag curvature and find its minimal value, thus provide the proofs of the above theorems.

2. The Randers metric and its navigation data

Suppose $\hat{F} = \hat{\alpha} + \hat{\beta}$ is a Randers norm on V. Without loss of generality, we may assume that $\hat{F}$ has navigation data $(\hat{h}, \hat{W})$, where $\hat{h}(y) = |y|$ is the standard Euclidean norm on V and $\hat{W}$ is a fixed vector in V with $\hat{h}(\hat{W}) = |\hat{W}| = \epsilon \in (0, 1)$. Let S^n be the standard Euclidean sphere

$$S^n = \{x \in V \mid \hat{h}(x) = 1\}.$$

Then the unit sphere Σ in $(V, \hat{F})$ is nothing but S^n shifted $\hat{W}$. Since $(V, \hat{F})$ is invariant by translations, the induced metric on Σ is isometric to the one induced on S^n. So we will work on S^n other than Σ.

Now we try to describe the induced Randers metric on S^n. At each point $x \in S^n$, we have an induced indicatrix $I_x = \{y \in T_x S^n \mid \hat{F}(y) = 1\}$. To obtain I_x, one can first construct the unit Euclidean sphere with center x, then shift it by the vector $\hat{W}$, and finally cut it by the tangent hyperplane $T_x S^n$. Notice that the hyperplane $T_x S^n$ has unit normal x (with respect to the Euclidean metric $\hat{h}$), so I_x is a sphere of radius $\sqrt{1 - \langle \hat{W}, x\rangle^2}$, whose center is $\hat{W} - \langle \hat{W}, x\rangle x \in T_x S^n$. Thus we have proved

Proposition 2.1. *The induced Randers metric on S^n has navigation data (h, W), where*

$$h(x, y) = \frac{|y|}{\sqrt{1 - \langle \hat{W}, x\rangle^2}}, \quad W(x) = \hat{W} - \langle \hat{W}, x\rangle x.$$

Notice that in the above proposition, $|y|$ is the standard Riemannian metric on S^n with constant sectional curvature $+1$. Write $\hat{W} = \epsilon a$ with $a \in S^n$, then we may assume $\langle \hat{W}, x\rangle = \epsilon \cos r$, where r is the spherical distance between a and x. Notice that $W(x)$ belongs to the plane $\mathrm{span}\{a, x\}$, so it is proportional to $\frac{\partial}{\partial r}$. Together with the fact that $|W(x)| = \epsilon \sin r$, we have

$$h = \frac{\sqrt{\mathrm{d}\, r^2 + \sin^2 r \ \mathrm{d}\, \sigma^2}}{\sqrt{1 - \epsilon^2 \cos^2 r}}, \quad W = -\epsilon \sin r \frac{\partial}{\partial r},$$

where $\mathrm{d}\, \sigma$ is the standard Riemannian metric on S^{n-1}. By using the well known routine we obtain

Corollary 2.2. *The induced Randers metric on S^n is given by $(1 - \epsilon^2)^{-1} \cdot F$, where $F = \alpha + \beta$ and*

$$\alpha = \sqrt{(1 - \epsilon^2 \cos^2 r)\, \mathrm{d}\, r^2 + (1 - \epsilon^2) \sin^2 r \ \mathrm{d}\, \sigma^2}, \quad \beta = \epsilon \sin r \ \mathrm{d}\, r. \tag{2.1}$$

Consequently the flag curvature of the induced Randers metric is $(1 - \epsilon^2)^2$ times that of F.

A glance at the above expression shows that β is closed. This will be crucial in our computation of flag curvature.

3. Computation of flag curvature

The closeness of β implies that F is projectively related to α. To establish a relation between flag curvatures of F and α, we shall use the following theorem.

Theorem 3.1. *Let F and $\bar{F}$ be two projectively related Finsler metrics on a manifold M, then their Riemann curvature tensors R_y and $\bar{R}_y$ satisfy*

$$R_i^j = \bar{R}_i^k h_k^j + (P^2 - \dot{P})h_i^j,$$

where $h_i^j = \delta_i^j - F_{y^i}y^j/F$ is the angular metric, $P = \dot{F}/(2F)$ and the over dot denotes the action of the $\bar{F}$-spray, i.e., $\dot{P} = P_{x^i}y^i - 2\bar{G}^i P_{y^i}$.

We believe that this theorem is known to P. Foulon since an equivalent form of this theorem appears in [5].

Now we continue to discuss the Randers metric $F = \alpha + \beta$. Notice that the angular metrics of F and α satisfy

$$h_{ij} = FF_{y^iy^j} = F\alpha_{y^iy^j} = \frac{F}{\alpha}\bar{h}_{ij}.$$

By the above theorem we have

$$\begin{aligned}
K_F(y\wedge v, y) =& \frac{h_{jl}R_i^j v^i v^l}{F^2 h_{il}v^i v^l}\\
=& \frac{h_{jl}\bar{R}_i^k h_k^j v^i v^l}{F^2 h_{il}v^i v^l} + \frac{1}{F^2}(P^2 - \dot{P})\\
=& \frac{\bar{h}_{jl}\bar{R}_i^k h_k^j v^i v^l}{F^2 \bar{h}_{il}v^i v^l} + \frac{1}{F^2}(P^2 - \dot{P})\\
=& \frac{\bar{h}_{lk}\bar{R}_i^k v^i v^l}{F^2 \bar{h}_{il}v^i v^l} + \frac{1}{F^2}(P^2 - \dot{P})\\
=& K_\alpha(y\wedge v, y) + \frac{1}{F^2}(P^2 - \dot{P}).
\end{aligned}$$

Thus we have proved the following.

Theorem 3.2. *Let $F = \alpha + \beta$ be a Randers metric where β is a closed one form. Then the flag curvatures of F and α are related by*

$$K_F(y\wedge v, y) = K_\alpha(y\wedge v) + \frac{1}{F^2}(P^2 - \dot{P}),$$

where $P = \dot{F}/(2F)$ and the over dot denotes the action of the α-spray.

Remark 3.3. *This theorem can also be proved by using Bao-Robles-Shen's formula [2, 3]. The method above uses Foulon's formula [5] that relates the Jacobi endomorphisms of two projectively related Finsler metrics. The transition from Jacobi endomorphism to flag curvature is due to the fact that the angular metrics of F and α only differ by a scalar multiple. So actually this theorem holds even if α is not a Riemannian metric.*

According to the above result, the flag curvature K_F is just the sectional curvature K_α plus some correction terms. Both the Riemann curvature tensor of α and the correction terms depend on the Riemann connection of α, so we will compute the Riemann connection first.

3.1. Riemann connection of α

In standard notation, the Riemannian metric α is given by $\psi^2 \,\mathrm{d}\, r^2 + \phi^2 \,\mathrm{d}\,\sigma^2$, where

$$\psi = \sqrt{1 - \epsilon^2\cos^2 r}, \quad \phi = \sqrt{1-\epsilon^2}\sin r.$$

Let $\{\theta_i\}_{i=2}^n$ be a local orthonormal coframe field on S^{n-1} so that

$$\mathrm{d}\,\sigma^2 = (\theta_2)^2 + \cdots + (\theta_n)^2.$$

Suppose the connection forms on S^{n-1} are $\{\theta_{ij}\}$, then they satisfy the following structure equations

$$\mathrm{d}\,\theta_i = \theta_j \wedge \theta_{ji}, \quad \theta_{ij} + \theta_{ji} = 0.$$

Moreover, since $\mathrm{d}\,\sigma^2$ has constant sectional curvature $+1$, we have

$$\mathrm{d}\,\theta_{ij} - \theta_{ik} \wedge \theta_{kj} = -\theta_i \wedge \theta_j.$$

Here and after, the indices i, j, k, $\cdots$ will always be in the range $\{2, 3, \cdots, n\}$.

Let $\omega_1 = \psi\,\mathrm{d}\ r$, $\omega_i = \phi\,\theta_i$, $2 \leq i \leq n$, then $\{\omega_1, \omega_i\}$ is a local orthonormal coframe field on S^n. Direct computation yields

$$\begin{aligned}
\mathrm{d}\,\omega_1 &= 0,\\
\mathrm{d}\ \omega_i &= \phi'\ \mathrm{d}\,r \wedge \theta_i + \phi\ \mathrm{d}\,\theta_i\\
&= \phi'/\psi\,\omega_1 \wedge \theta_i + \phi\,\theta_j \wedge \theta_{ji}\\
&= \phi'/\psi\,\omega_1 \wedge \theta_i + \omega_j \wedge \theta_{ji}.
\end{aligned}$$

It follows that the one forms

$$\omega_{1i} = -\omega_{i1} = \phi'/\psi \cdot \theta_i, \quad \omega_{ji} = \theta_{ji}$$

will satisfy the structure equations

$$\mathrm{d}\,\omega_1 = \omega_i \wedge \omega_{i1}, \quad \mathrm{d}\,\omega_i = \omega_1 \wedge \omega_{1i} + \omega_j \wedge \omega_{ji}.$$

So, the 1-forms $\{\omega_{1i}, \omega_{ji}\}$ are connection forms of α.

3.2. Riemann curvature tensor of α

The curvatures of α are encoded in the curvature forms $\{\Omega_{1i}, \Omega_{ij}\}$. We have

$$\begin{aligned}
\Omega_{1i} &= \mathrm{d}\,\omega_{1i} - \omega_{1j} \wedge \omega_{ji} = \mathrm{d}(\phi'/\psi\,\theta_i) - \phi'/\psi\,\theta_j \wedge \theta_{ji}\\
&= (\phi'/\psi)'\ \mathrm{d}\,r \wedge \theta_i = (\phi'/\psi)'/(\psi\phi)\,\omega_1 \wedge \omega_i,\\
\Omega_{ij} &= \mathrm{d}\,\omega_{ij} - \omega_{i1} \wedge \omega_{1j} - \omega_{ik} \wedge \omega_{kj}\\
&= \mathrm{d}\,\theta_{ij} + (\phi'/\psi)^2\theta_i \wedge \theta_j - \theta_{ik} \wedge \theta_{kj}\\
&= \big((\phi'/\psi)^2 - 1\big)\theta_i \wedge \theta_j = \big((\phi'/\psi)^2 - 1\big)/\phi^2\,\omega_i \wedge \omega_j.
\end{aligned}$$

Consequently, the nonzero components of the Riemann curvature tensor are

$$\begin{aligned}
R_{1i1i} = &R_{i1i1} = -R_{1ii1} = -R_{i11i} = \frac{(\phi'/\psi)'}{\psi\phi} = -\frac{1}{\psi^4},\\
R_{ijij} = &R_{jiji} = -R_{ijji} = -R_{jiij} = \frac{(\phi'/\psi)^2 - 1}{\phi^2} = -\frac{1}{(1-\epsilon^2)\psi^2}, \quad i \neq j,
\end{aligned}$$

Now let $y = y^1 e_1 + y^i e_i$, where $\{e_1, e_i\}$ is the dual local frame field of $\{\omega_1, \omega_i\}$. Then we have $\alpha = \sqrt{(y^1)^2 + (y^2)^2 + \cdots + (y^n)^2}$ and

$$\begin{aligned}
R_{11} &= \sum_{i=2}^{n} R_{i11i} y^i y^i = (\alpha^2 - (y^1)^2)/\psi^4,\\
R_{1i} &= R_{i1} = R_{i1i1} y^i y^1 = -y^1 y^i/\psi^4,\\
R_{ii} &= R_{1ii1} y^1 y^1 + \sum_{j\neq i} R_{jiij} y^j y^j = (y^1)^2/\psi^4 + (\alpha^2 - (y^1)^2) - (y^i)^2)/((1-\epsilon^2)\psi^2),\\
R_{ij} &= R_{ji} = R_{jiji} y^j y^i = -y^j y^i/((1-\epsilon^2)\psi^2), \quad i \neq j.
\end{aligned}$$

The last two lines can be summarized as

$$R_{ij} = \frac{1}{\psi^4}(y^1)^2\delta_{ij} + \frac{1}{(1-\epsilon^2)\psi^2}\big((\alpha^2 - (y^1)^2)\delta_{ij} - y^i y^j\big).$$

Proposition 3.4. *Let α be the Riemannian metric given by (2.1), then the eigenvalues of the Riemann curvature tensor $\bar{R}_y$ are*

$$\begin{aligned}
&\lambda_0 = 0, \quad \lambda_1 = \alpha^2\psi^{-4},\\
&\lambda_2 = (y^1)^2\psi^{-4} + (\alpha^2 - (y^1)^2)(1-\epsilon^2)^{-1}\psi^{-2},
\end{aligned}$$

where λ_2 has multiplicity $n - 2$.

Proof. We say that y is in general position if $\alpha^2 \neq (y^1)^2$. For such y, the components $y^2, \cdots, y^n$ are not simultaneously zero.

Now assume y is in general position. It is well known that $\bar{R}_y(y) = 0$, so $\bar{R}_y$ has eigenvalue 0 with eigenvector y.

Let $v = v^1 e_1 + v^i e_i$ with $v^1 = \alpha^2 - (y^1)^2$, $v^i = -y^1 y^i$, then we have

$$R_{11}v^1 + R_{1i}v^i = \alpha^2\psi^{-4} \cdot (\alpha^2 - (y^1)^2),$$
$$R_{i1}v^1 + R_{ij}v^j = -\alpha^2\psi^{-4}y^1y^i.$$

It follows that $\bar{R}_y(v) = \lambda_1 \cdot v$, i.e., $\lambda_1 = \alpha^2\psi^{-4}$ is an eigenvalue of $\bar{R}_y$ with eigenvector v.

Now fix a pair of indices i, j with $i \neq j$ and let $w = -y^j e_i + y^i e_j$, then one can check that $\bar{R}_y(w) = \lambda_2 w$. For all possible choices of i, j, such vector w forms a linear space of dimension $n-2$. Thus λ_2 is an eigenvalue of multiplicity $n-2$.

Thus, when y is in general position, the proposition is proved. If y is not in general position, the result also follows since $\bar{R}_y$ is continuous in y. Indeed, when $\alpha^2 = (y^1)^2$, the eigenvalues λ_1 and λ_2 are equal and have multiplicity $n-1$. □

Corollary 3.5. *Fix a nonzero vector $y \in T_xS^n$. Let π be a plane in T_xS^n that contains y, then the sectional curvature $K_\alpha(\pi)$ satisfies*

$$\psi^{-4} \leq K_\alpha(\pi) \leq s^2\psi^{-4} + (1-s^2)(1-\epsilon^2)^{-1}\psi^{-2},$$

where $s = y^1/\alpha$.

Proof. Suppose π is spanned by y and $v = v^1 e_1 + v^i e_i$. Without loss of generality we may assume that y and v are orthogonal. We have

$$K_\alpha(\pi) = \frac{R_{ij}v^iv^j}{\alpha^2\delta_{ij}v^iv^j}.$$

Thus $K_\alpha(\pi)$ attains its minimal and maximal values at the nonzero eigenvalues of $\bar{R}_y/\alpha^2$. □

3.3. The correction terms

To treat the correction terms, we need to find the α-spray. By definition, α-spray is the unique vector field $\bar{S}$ on TS^n satisfying

$$\mathrm{d}\,\omega(\bar{S},\,\cdot\,) = -\,\mathrm{d}\,\alpha^2/2,$$

where $\omega = y^1\omega_1 + y^i\omega_i$ is the Hilbert form. Notice that

$$\begin{aligned}\mathrm{d}\,\omega &= \mathrm{d}\,y^1 \wedge \omega_1 + \mathrm{d}\,y^i \wedge \omega_i + y^i\,\mathrm{d}\,\omega_i \\ &= \mathrm{d}\,y^1 \wedge \omega_1 + \mathrm{d}\,y^i \wedge \omega_i + y^i(\omega_1 \wedge \omega_{1i} + \omega_j \wedge \theta_{ji}),\end{aligned}$$

we have

$$\bar{S} = y^1 e_1 + y^i e_i - 2G^1\frac{\partial}{\partial y^1} - 2G^i\frac{\partial}{\partial y^i},$$

where the coefficient G^1 is given by

$$G^1 = -\frac{1}{2}\sum_i (y^i)^2\omega_{1i}(e_i) = -\frac{1}{2}(\alpha^2 - (y^1)^2)\phi'/(\psi\phi).$$

If we introduce the abbreviation $s = y^1/\alpha$, then G^1 can also be written as

$$G^1 = -\frac{1}{2}\alpha^2(1-s^2)\phi'/(\psi\phi).$$

The other coefficients G^i will not be used, because we can express the quantity P as an expression of α, r, s, and we have the following lemma.

Lemma 3.6. *Suppose f is an expression of α, r, s, then we have*

$$\bar{S}(f) = \alpha \cdot \big((s/\psi) \cdot f_r + (1-s^2)\phi'/(\psi\phi) \cdot f_s\big).$$

Proof. Notice that $\bar{S}(\alpha) = 0$, it remains to prove that $\bar{S}(s) = -2G^1/\alpha$ and $\bar{S}(r) = \alpha s/\psi$. They are straightforward. □

Now, the one form β can be written as $by^1 = b\alpha s$, where

$$b = \epsilon \sin r \psi^{-1}.$$

So we have $F = (1 + bs)\alpha$. By using the above lemma we obtain

$$\begin{aligned}\dot{F} = \bar{S}(F) = \alpha \bar{S}(bs) &= \alpha^2 \cdot \big(s^2 b'/\psi + b(1-s^2)\phi'/(\psi\phi)\big)\\ &= \alpha^2 \epsilon \cos r (\psi^2 - \mu^2)\psi^{-4},\end{aligned}$$

where $\mu = \epsilon s \sin r$. Further differentiation yields

$$\ddot{F} = \bar{S}(\dot{F}) = -\alpha^3 \mu (4 - 3\psi^2)(\psi^2 - \mu^2)\psi^{-7}.$$

Since $P = \dot{F}/(2F)$, we find that the correction term is

$$\frac{1}{F^2}(P^2 - \dot{P}) = \frac{3\dot{F}^2 - 2F\ddot{F}}{4F^4}.$$

Substituting the previous results into the above expression, we conclude that the correction term equals

$$\frac{(\psi - \mu)\big((5 - 3\psi^2)\mu^2 + 2(4 - 3\psi^2)\psi\mu + 3\psi^2 - 3\psi^4\big)}{4\psi^4(\psi + \mu)^3}.$$

By adding the above correction term to Corrollary 3.5, we get a preliminary bound of the flag curvatures of F.

4. Flag curvature lower bound

In this section we will try to find the minimal value of K_F.

As Corollary 3.5 shows, $K_\alpha \geq \psi^{-4}$. So we have

$$\begin{aligned}K_F \geq &\frac{1}{\psi^4} + \frac{(\psi - \mu)\big((5 - 3\psi^2)\mu^2 + 2(4 - 3\psi^2)\psi\mu + 3\psi^2 - 3\psi^4\big)}{4\psi^4(\psi + \mu)^3}\\ = &\frac{(3\psi^2 - 1)\mu^2 + 10\psi\mu - 3\psi^4 + 7\psi^2}{4\psi^4(\psi + \mu)^2}.\end{aligned}$$

Denote the right hand side by K_1. Recall that

$$\psi = \sqrt{1 - \epsilon^2 \cos^2 r}, \quad \mu = \epsilon s \sin r, \quad s = y^1/\alpha.$$

The variables ψ and μ should satisfy the following constraints

$$\psi^2 - \mu^2 \geq 1 - \epsilon^2, \quad \sqrt{1 - \epsilon^2} \leq \psi \leq 1. \tag{4.1}$$

Thus, to find the minimal value of K_F, we only need to minimize the two variable function K_1 subject to the above constraints.

However, if we are only interested in the sign of K_1, then the denominator of K_1 has no influence at all. It remains to consider the numerator

$$N_1 = (3\psi^2 - 1)\mu^2 + 10\psi\mu - 3\psi^4 + 7\psi^2.$$

Since $|\mu| < \psi \leq 1$, we have

$$\frac{\partial N_1}{\partial \mu} = 2(3\psi^2 - 1)\mu + 10\psi > 2\psi(5 - |3\psi^2 - 1|) > 0.$$

It follows that N_1 is increasing in the variable μ. For each fixed ψ, the minimal value of N_1 is attained at $\mu_0 = -\sqrt{\psi^2 - 1 + \epsilon^2}$ (this is equvalent to $s = -1$). Thus we are leading to the function

$$\begin{aligned}N_2 &= (3\psi^2 - 1)(\psi^2 - 1 + \epsilon^2) - 10\psi\sqrt{\psi^2 - 1 + \epsilon^2} - 3\psi^4 + 7\psi^2\\ &= -10\psi\sqrt{\psi^2 - 1 + \epsilon^2} + (3\epsilon^2 + 3)\psi^2 + (1 - \epsilon^2).\end{aligned}$$

Equivalently, we can consider the rationalized version

$$\begin{aligned} N_3 &= -100\psi^2(\psi^2 - 1 + \epsilon^2) + ((3\epsilon^2 + 3)\psi^2 + (1 - \epsilon^2))^2 \\ &= -(91 - 18\epsilon^2 - 9\epsilon^4)\psi^4 + (106 - 100\epsilon^2 - 6\epsilon^4)\psi^2 + (1 - \epsilon^2)^2. \end{aligned}$$

Make a change of variables

$$t = \psi^2, \quad e = \epsilon^2,$$

then we have

$$N_3 = -(91 - 18e - 9e^2)t^2 + (106 - 100e - 6e^2)t + (1 - e)^2,$$

where t and e satisfy the relations $1 - e \le t \le 1$ and $0 < e < 1$.

Since the leading coefficient $-(91 - 18e - 9e^2)$ is negative, N_3, as a quadratic function of t, must attain its minimal value at $1 - e$ or 1. Notice that the values of N_3 at $t = 1 - e$ and $t = 1$ are $(e - 1)^2(3e + 4)^2$ and $4(e^2 - 21e + 4)$, respectively. Moreover,

$$(e - 1)^2(3e + 4)^2 - 4(e^2 - 21e + 4) = e(76 - 27e + 6e^2 + 9e^3) > 0.$$

We conclude that the minimal value of N_3 is $4(e^2 - 21e + 4)$, which is attained at $t = 1$. This minimal value is positive if and only if $e < (21 - 5\sqrt{17})/2 = (5 - \sqrt{17})^2/4$.

Let $\epsilon_0 = (5 - \sqrt{17})/2 \approx 0.43845$, then the above result can be summarized as the following theorem.

Theorem 4.1. *Let $\hat{F} = \hat{\alpha} + \hat{\beta}$ be a Randers norm on an $n+1$ dimensional vector space V. Let $\epsilon := |\hat{\beta}|_{\hat{\alpha}}$ and let Σ be the unit sphere in V. If $\epsilon < \epsilon_0$, then Σ has positive flag curvature everywhere; if $\epsilon = \epsilon_0$, then Σ is non-negatively curved and the zero flag curvature only appears at the equator of Σ; if $\epsilon > \epsilon_0$, then there are flags on the equator of Σ with negative flag curvature.*

Remark 4.2. *One may wonder if there is an ϵ such that Σ is negatively curved. This is impossible because according to the classical Cartan-Hadamard's theorem, S^n cannot carry a Finsler metric with everywhere non-positive flag curvature since its universal cover is not $\mathbb{R}^n$. The above analysis provieds another interpretation to this fact, because every flag at the pole $r = 0$ or $r = \pi$ has positive curvature.*

Motivated by the above result, it is natural to conjecture that the minimal value of K_1 is attained at $\psi = 1$ and $\mu = -\epsilon$. This is indeed the case. To give a detailed proof of this fact, we shall need the following

Lemma 4.3. *Notations as above, then we have*

$$K_1 \ge \frac{\epsilon^2 - 5\epsilon + 2}{2(\epsilon - 1)^2}, \tag{4.2}$$

with equality holds if and only if $\psi = 1$ and $\mu = -\epsilon$.

Proof. We may view K_1 as a two-variable function whose domain is given by (4.1). To find the minimal value of K_1, we first look at the interior extreme points, namely, those satisfying $(K_1)_\psi = (K_1)_\mu = 0$. Direct computation shows that

$$\begin{aligned} (K_1)_\mu &= \frac{(3\psi^2 - 2)\psi - 3(2 - \psi^2)\mu}{2\psi^3(\psi + \mu)^3}, \\ (K_1)_\psi &= \frac{-(3\psi^2 - 2)\mu^3 - 6(\psi^2 + 2)\psi\mu^2 - 32\psi^2\mu + (3\psi^2 - 14)\psi^3}{2\psi^5(\psi + \mu)^3}. \end{aligned}$$

The solutions of the system $(K_1)_\mu = (K_1)_\psi = 0$ are $(\mu, \psi) = (0, 0)$, $(-2\sqrt{6}, 2\sqrt{6}/3)$ or $(2\sqrt{6}, -2\sqrt{6}/3)$. None of these points are interior points. So the minimal value of K_1 must be attained at the boundary.

If $\psi = \sqrt{1 - \mu^2}$, then $\mu = 0$, in this case $K_1 = \frac{3\epsilon^2+4}{4(1-\epsilon^2)^2} > 0$.

If $\psi = 1$, then $\mu = \pm\epsilon$, in this case $K_1 = \frac{\epsilon^2 \pm 5\epsilon + 2}{2(1+\epsilon)^2}$.

It is easy to show that $\frac{3\epsilon^2+4}{4(1-\epsilon^2)^2} \ge \frac{\epsilon^2 - 5\epsilon + 2}{2(1+\epsilon)^2}$, thus the minimal value of K_1 is attained at $\psi = 1$ and $\mu = -\epsilon$. □

As a consequence of the inequality (4.2), the flag curvature lower bound of the induced metric on S^n is $\frac{1}{2}(\epsilon + 1)^2(\epsilon^2 - 5\epsilon + 2)$, thus completing the proof of Theorem 1.2.

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