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Rank inequality in homogeneous Finsler geometry

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ABSTRACT: This is a survey on some recent progress in homogeneous Finsler geometry. Three topics are discussed, the classification of positively curved homogeneous Finsler spaces, the geometric and topological properties of homogeneous Finsler spaces satisfying $K \geq 0$ and the (FP) condition, and the orbit number of prime closed geodesics in a compact homogeneous Finsler manifold. These topics share the same similarity that the same rank inequality, i.e., $\mathrm{rank}G \leq \mathrm{rank}H + 1$ for G/H with compact G and H, plays an important role. In this survey, we discuss in each topic how the rank inequality is proved, explain its importance, and summarize some relevant results.

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(*Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration*)

1. Introduction

The rank inequality, i.e., $\mathrm{rank}G \leq \mathrm{rank}H + 1$ for a smooth coset space G/H with compact G and H, was originated from the classification of positively curved Riemannian homogeneous spaces [3, 6, 8, 52]. It plays an important role in this classification project by simplifying and systemizing the case-by-case discussion. The importance of rank inequality extends to homogeneous Finsler geometry. Three topics are discussed to justify our viewpoint.

The first is the classification of positively curved homogeneous Finsler spaces. After the proof of rank inequality in this context by S. Deng and Z. Hu [21], there had been big progress in this classification project. More details and references will be given in Section 5.2. See also the survey [24] and the references therein.

The second is the *compactness and rank inequality conjecture* for a homogeneous Finsler manifold satisfying $K \geq 0$ (i.e., the non-negatively flag curvature property) and the (FP) condition (see Conjecture 6.2). The (FP) condition (see Definition 6.1) introduced in [56] and [64] provides new thoughts for studying the positive curvature problem in Finsler geometry. The combination of $K \geq 0$ and the (FP) condition is a reasonable and interesting approximation for the positive flag curvature property [64]. Solving Conjecture 6.2 opens the gate to a new classification project [59, 68].

The last topic concerns the orbit number for prime closed geodesics on a closed Finsler manifold. In [58], the following *closed geodesic orbit number conjecture* is announced: any closed Finsler manifold, if it admits nontrivial

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continuous isometric actions, must have two orbits of prime closed geodesics, unless it is isometric to a rank-one Riemannian symmetric space (see Conjecture 7.2). To explore this conjecture, a closed Finsler manifold (M, F) with $\dim I(M, F) > 0$ and only orbit of prime closed geodesics has been studied. It has been proved recently in [58] that in this (M, F), there exists an orbit $G \cdot x = G/H$ of $G = I_0(M, F)$ containing all closed geodesics, which satisfies the rank inequality $\mathrm{rank}G \leq \mathrm{rank}H + 1$. Using this rank inequality and the relevant algebraic setup for case-by-case discussion, we proved the closed geodesic orbit number conjecture for compact homogeneous Finsler spaces which are either even-dimensional or reversible [58].

To summarize, we expect to see more progress in homogeneous Finsler geometry which are inspired or influenced by this amazing rank inequality.

This survey is organized as follows. In Section 2, we summarize some preliminaries in general and homogeneous Finsler geometry. In Section 3, we introduce the algebraic setup relevant to the rank inequality. In Section 4, we switch to the classification of positive curvature in homogeneous Finsler geometry. In Section 5, we discuss the rank inequality in the classification of positively curved homogeneous Finsler spaces. In Section 6, we discuss the (FP) condition, examples and the compactness, and rank inequality conjecture. In Section 7, we reveal the importance of rank inequality for studying the closed geodesic orbit number conjecture.

2. Preliminaries

In this section, we summarize some fundamental knowledge in general Finsler geometry [11, 48], homogeneous Finsler geometry [19], and Lie theory [33]. In later discussion, we only consider connected smooth manifolds which dimensions are bigger than 1.

2.1. Finsler metric and Minkowski norm

A *Finsler metric* on a smooth manifold M is a continuous function $F : TM \to [0, +\infty)$ satisfying the following properties:

1. F is positive and smooth when restricted to $TM\backslash 0$;
2. $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda \geq 0$;
3. With respect to any standard local coordinates (x^i, y^i), i.e., $x = (x^i) \in M$, $y = y^i\partial_{x^i} \in T_xM$, the Hessian matrix $(g_{ij}(x, y)) = (\frac{1}{2}[F^2(x, y)]_{y^iy^j})$ is positive definite for any $y \in T_xM\backslash\{0\}$.

We call (M, F) a *Finsler space* or a *Finsler manifold.*

The restriction of F to each tangent space is called a *Minkowski norm.* Minkowski norm can also be abstractly defined on any finite dimensional real vector space using similar conditions as (1)-(3) above.

A Finsler metric F is called reversible if $F(x, y) = F(x, -y)$ is always satisfied. The Hessian matrix $(g_{ij}(x, y))$ defines an inner product $g_y(\cdot, \cdot)$ parametrized by the nonzero base vector y, i.e.,

$$g_y(u, v) = g_{ij}(x, y)u^iv^j = \frac{1}{2}[F^2(y + su + tv)]_{st}|_{s=t=0}.$$

Sometimes we call the Hessian $(g_{ij}(x, y))$ or the inner product $g_y(\cdot, \cdot)$ the *fundamental tensor.*

Riemannian metrics are a special class of reversible Finsler metrics. They are characterized by the property that the fundamental tensor does not depend on the base vector y. *Randers metrics*, which have the form $F = \alpha + \beta$, in which α is a Riemannian metric and β is a one-form, are the most simple and the most important irreversible Finsler metrics [45]. (α, β)-*metrics* have the form $F = \alpha\phi(\beta/\alpha)$, in which $\phi(\cdot)$ is a positive smooth one-variable function, and α, β are similar to those for Randers metrics [41].

2.2. Geodesic, geodesic spray and S-curvature

Using the locally minimizing principle for the arc length functional $l(c) = \int_a^b F(\dot{c}(t))dt$ for a piecewise smooth curve $c(t) : [a, b] \to M$, a *geodesic* can be defined on the Finsler manifold (M, F). Unless otherwise specified, a geodesic is assumed to have a positive constant speed.

A geodesic $c(t)$ can be lifted to the curve $(c(t), \dot{c}(t))$ in $TM\backslash 0$ and then described as the integral curve of the *geodesic spray*

$$\mathbf{G} = y^i\partial_{x^i} - 2\mathbf{G}^i\partial_{y^i},$$

in which

$$\mathbf{G}^i = \tfrac{1}{4}g^{il}([F^2]_{x^ky^l}y^k - [F^2]_{x^l}).$$

Though the geodesic spray is presented by standard local coordinates, it is in fact globally and smoothly defined on $TM\backslash 0$. So a geodesic $c(t)$ can also be determined by standard local coordinates as the solution of the following ODE system

$$\ddot{c}^i(t) + 2\mathbf{G}^i(c(t), \dot{c}(t)) = 0, \quad \forall i.$$

The *S-curvature* was discovered by Z. Shen when he studied the volume comparison in Finsler geometry [47]. Here we only introduce the S-curvature with respect to the B.H. volume form $d\mu^F_{\mathrm{BH}} = \sigma(x)dx^1 \cdots dx^n$ on (M^n, F), in which

$$\sigma(x) = \frac{\mathrm{Vol}(\{(y^i) \in \mathbb{R}^n | \sum (y^i)^2 \leq 1\})}{\mathrm{Vol}(\{y = y^i \partial_{x^i} | F(x, y) \leq 1\})},$$

where $\mathrm{Vol}(\cdot)$ is the standard Euclidean measure in $\mathbb{R}^n$. The S-curvature $S(x, y) : TM\backslash 0 \to \mathbb{R}$ of a Finsler manifold (M, F) is the directional derivative of the distortion function

$$\tau(x, y) = \ln\left(\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}\right)$$

in the direction of $\mathbf{G}(x, y)$.

2.3. Riemann curvature and flag curvature

Riemann curvature $R_y : T_xM \to T_xM$ for any nonzero $y \in T_xM$ is a $g_y(\cdot, \cdot)$-self-adjoint linear operator, which appears in the Jacobi equation for the geodesic variations. Using standard local coordinates, it can be presented as $R_y = R^i_k \partial_{x^i} \otimes dx^k$, in which

$$R^i_k(y) = 2[\mathbf{G}^i]_{x^k} - y^j[\mathbf{G}^i]_{x^j y^k} + 2\mathbf{G}^j[\mathbf{G}^i]_{y^j y^k} - [\mathbf{G}^i]^j_y[\mathbf{G}^j]_{y^k}.$$

Consider a triple $(x, y, \mathbf{P})$ with the point $x \in M$, the nonzero vector (i.e., the *flagpole*) $y \in T_xM$ and the tangent plane (i.e., the *flag*) $\mathbf{P} = \mathrm{span}\{y, u\} \subset T_xM$. Then the *flag curvature* $K(x, y, \mathbf{P}) = K(x, y, y \wedge u)$ is defined as

$$K(x, y, \mathbf{P}) = \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$

Generally speaking, flag curvature depends on the flag as well as on the flagpole. However, when F is Riemannian, it is reduced to sectional curvature which is only relevant to the flag.

2.4. Homogeneous Finsler geometry

The *isometry group* $I(M, F)$ for a Finsler manifold is the group of all diffeomorphisms $\rho : M \to M$ satisfying $\rho^* F = F$. It is a Lie transformation group [20]. Its Lie algebra can be identified as the space of Killing vector fields on (M, F), i.e., each $v \in \mathrm{Lie}(I(M, F))$ induces the Killing vector field

$$V(x) = \frac{d}{dt}|_{t=0}(\exp tv \cdot x).$$

A Finsler manifold (M, F) is called *homogeneous* if its isometry group $I(M, F)$ acts transitively on M. A homogeneous Finsler space (M, F) can be presented as $(G/H, F)$ for any Lie subgroup $G \subset I(M, F)$ which acts transitively on M. As M is assumed connectedness, we may require G to be a connected Lie subgroup of the *connected isometry group* $I_0(M, F)$ (which is the identity component of $I(M, F)$). The presentation G/H for a homogeneous Finsler space may not be unique. For example, the homogeneous spheres have been classified as following [9, 43],

$$\begin{aligned}
&S^n = SO(n+1)/SO(n), \quad S^{2n-1} = SU(n)/SU(n-1) = U(n)/U(n-1),\\
&S^{4n-1} = Sp(n)/Sp(n-1) = Sp(n)U(1)/Sp(n-1)U(1) = Sp(n)Sp(1)/Sp(n-1)Sp(1),\\
&S^6 = G_2/SU(3), \quad S^7 = Spin(7)/G_2, \quad S^{15} = Spin(9)/Spin(7).
\end{aligned}$$

In the presentation $M = G/H$ for the homogeneous Finsler manifold (M, F), H is the isotropy subgroup at $o = eH$. It is a compactly imbedded subgroup of G [44], so in the Lie algebra level, we have a *reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \tag{2.1}$$

for G/H, in which $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h} = \mathrm{Lie}(H)$ in $\mathfrak{g} = \mathrm{Lie}(G)$. The subspace $\mathfrak{m}$ can be canonically viewed as the tangent space $T_o(G/H)$ at $o = eH$, with the $\mathrm{Ad}(H)$-action identified with the isotropic action.

There is a subtlety that the compactly imbedded subgroup H may not be compact itself, though it has a compact Lie algebra. For simplicity, you may further assume G to be closed in $I_0(M,F)$, then H is compact and the existence of the reductive decomposition (2.1) is more obvious [10].

The reductive decomposition for the homogeneous Finsler space (M,F) may not be unique. Since the restriction $B_{\mathfrak{g}}|_{\mathfrak{h}\times\mathfrak{h}}$ of the Killing form $B_{\mathfrak{g}}$ to $\mathfrak{h}$ is nondegenerate [44], the *canonical reductive decomposition* is the unique one which is $B_{\mathfrak{g}}$-orthogonal. When G or $\mathfrak{g}$ is compact, the canonical reductive decomposition can be equivalently determined by its orthogonality with respect to certain $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$.

Reductive decomposition is a fundamental technique in homogeneous Riemannian geometry and homogeneous Finsler geometry. Any homogeneous (i.e., G-invariant) Finsler metric F on G/H can be one-to-one determined by its restriction to $T_o(G/H)$, which is an $\mathrm{Ad}(H)$-invariant Minkowski norm [19] (for simplicity, we still use F to denote this Minkowski norm).

3. Algebraic setup for the rank inequality

Suppose G/H is coset space with compact G and H and satisfies $\mathrm{rank}G \leq \mathrm{rank}H + 1$, in which the *rank* is the dimension of a maximal torus subgroup. The rank inequality can help us simplify systemize the case-by-case discussion for G/H. The following algebraic setup is useful for classifying positively curved homogeneous Finsler spaces [62, 63], as well as for studying the closed geodesic orbit number for a compact homogeneous Finsler space [58].

Firstly, we have the canonical reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for G/H, which is orthogonal with respect to certain $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$.

Then we fix a *fundamental Cartan subalgebra* $\mathfrak{t}$ of $\mathfrak{g}$, i.e., $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{t}\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$. The root plane decompositions for $\mathfrak{g}$ and $\mathfrak{h}$ are for $\mathfrak{t}$ and $\mathfrak{t}\cap\mathfrak{h}$ respectively.

3.1. Even dimensional case

Assume $\mathrm{rank}G = \mathrm{rank}H$. In this case, $\dim G/H$ is even and we have $\mathfrak{t}\subset\mathfrak{h}$.

With respect to $\mathfrak{t}$, the root plane decompositions are the following

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha\in\Delta_{\mathfrak{g}}} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{h} = \mathfrak{t} + \sum_{\alpha\in\Delta_{\mathfrak{h}}} \mathfrak{h}_{\pm\alpha} = \mathfrak{t} + \sum_{\alpha\in\Delta_{\mathfrak{h}}} \mathfrak{g}_{\pm\alpha}, \quad \text{and} \quad \mathfrak{m} = \sum_{\alpha\notin\Delta_{\mathfrak{h}}} \mathfrak{g}_{\pm\alpha}.$$

Using the chosen bi-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$, we can identify the root system $\Delta_{\mathfrak{g}}$ as a subset in $\mathfrak{t}$. Using the restriction of $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{h}$, we can also identify the root system $\Delta_{\mathfrak{h}}$ as a subset in $\mathfrak{t}$. Furthermore, we have the relation $\Delta_{\mathfrak{h}}\subset\Delta_{\mathfrak{g}}$ between the two root systems. Each root plane $\mathfrak{h}_{\pm\alpha}$ of $\mathfrak{h}$ coincides with the root plane $\mathfrak{g}_{\pm\alpha}$ of $\mathfrak{g}$.

3.2. Odd dimensional case

Assume $\mathrm{rank}G = \mathrm{rank}H + 1$. In this case, $\dim G/H$ is odd and we have $\dim\mathfrak{t}\cap\mathfrak{m} = 1$.

We have the following root plane decompositions,

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha\in\Delta_{\mathfrak{g}}} \mathfrak{g}_{\pm\alpha}, = \sum_{\alpha'\in\mathfrak{t}\cap\mathfrak{h}} \hat{\mathfrak{g}}_{\pm\alpha'} \quad \mathfrak{h} = \mathfrak{t}\cap\mathfrak{h} + \sum_{\alpha'\in\Delta_{\mathfrak{h}}} \mathfrak{h}_{\pm\alpha'}, \quad \text{and} \quad \mathfrak{m} = \sum_{\alpha'\in\mathfrak{t}\cap\mathfrak{h}} \hat{\mathfrak{m}}_{\pm\alpha'}.$$

Here we still use $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ and its restriction to $\mathfrak{h}$ to identify $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{h}}$ as subsets in $\mathfrak{t}$ and $\mathfrak{t}\cap\mathfrak{h}$ respectively. Denote $\mathrm{pr}:\mathfrak{t}\to\mathfrak{t}\cap\mathfrak{h}$ the orthogonal projection. Then $\hat{\mathfrak{g}}_{\pm\alpha'}$ for $\alpha'\in\mathfrak{t}\cap\mathfrak{h}\backslash\{0\}$ is the sum of all root planes $\mathfrak{g}_{\pm\alpha}$ with $\mathrm{pr}(\alpha)=\alpha'$, $\hat{\mathfrak{g}}_0 = \mathfrak{t} + \sum_{\mathrm{pr}(\alpha)=0}\mathfrak{g}_{\pm\alpha}$, and $\hat{\mathfrak{m}}_{\pm\alpha'} = \hat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{m}$. Each $\hat{\mathfrak{g}}_{\pm\alpha'}$ is compatible with reductive decomposition (2.1), i.e.,

$$\hat{\mathfrak{g}}_{\pm\alpha'} = \hat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{h} + \hat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{m} = \hat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{h} + \hat{\mathfrak{m}}_{\pm\alpha'}.$$

When $\alpha'\in\Delta_{\mathfrak{h}}$, $\hat{\mathfrak{g}}_{\pm\alpha'}\cap\mathfrak{h}$ is the root plane $\mathfrak{h}_{\pm\alpha'}$ of $\mathfrak{h}$, otherwise it is just zero.

In $\hat{\mathfrak{g}}_0 = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}\cap\mathfrak{h}) = \mathfrak{t}\cap\mathfrak{h} + \hat{\mathfrak{m}}_0$, $\hat{\mathfrak{m}}_0 = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{t}\cap\mathfrak{h})\cap\mathfrak{m}$ is a Lie subalgebra. If $\Delta_{\mathfrak{g}}\cap\mathfrak{m}\neq\emptyset$, i.e., there exists a pair of roots $\pm\alpha$ of $\mathfrak{g}$ contained in the line $\mathfrak{t}\cap\mathfrak{m}$, $\hat{\mathfrak{m}}_0 = \mathfrak{t}\cap\mathfrak{m} + \mathfrak{g}_{\pm\alpha}$ is of type $\mathfrak{a}_1$. Otherwise, $\hat{\mathfrak{m}}_0 = \mathfrak{t}\cap\mathfrak{m}$ is Abelian.

3.3. Sorting G/H according to regularity

Finally, we sort all the cases of G/H according to the regularity of $\mathfrak{h}$ in $\mathfrak{g}$. We say the subalgebra $\mathfrak{h}$ is *regular* in $\mathfrak{g}$ (with respect to the chosen fundamental Cartan subalgebra $\mathfrak{t}$), when each root plane of $\mathfrak{h}$ with respect to $\mathfrak{t}\cap\mathfrak{h}$ is a root plane of $\mathfrak{g}$ with respect to $\mathfrak{t}$.

When $\mathrm{rank}G = \mathrm{rank}H$, $\mathfrak{h}$ is always regular in $\mathfrak{g}$. This case is relatively easy to handle.

When $\mathrm{rank}G = \mathrm{rank}H + 1$, G/H can be further sorted into three categories:

Category I. $\mathfrak{h}$ is regular in $\mathfrak{g}$;

Category II. $\mathfrak{h}$ is irregular in $\mathfrak{g}$, and there exist two roots α and β of $\mathfrak{g}$ from different simple ideals, such that $\mathrm{pr}(\alpha) = \mathrm{pr}(\beta) = \alpha'$ is a root of $\mathfrak{h}$;

Category III. $\mathfrak{h}$ is irregular in $\mathfrak{g}$, and there exist two roots of $\mathfrak{g}$ from the same simple ideals, such that $\mathrm{pr}(\alpha) = \mathrm{pr}(\beta) = \alpha'$ is a root of $\mathfrak{h}$.

Further discussion for G/H can be carried out case-by-case in each Category.

4. Rank inequality for $K > 0$ in homogeneous Riemannian geometry

4.1. Classification of positively curved Riemannian homogeneous spaces

A closed manifold with positive sectional curvature is a central topic in Riemannian geometry. On one hand, it has many intriguing properties, from the view points of differential topology and geometric analysis [71]. On the other hand, examples are relatively rare. The complete list of positively curved Riemannian homogeneous spaces (up to local isometries) was found during 1960's and 1970's [3, 8, 6, 52], i.e.,

1. compact rank-one Riemannian symmetric spaces

$$S^n = SO(n+1)/SO(n), \quad \mathbb{C}\mathrm{P}^n = SU(n+1)/S(U(n)U(1)),$$
$$\mathbb{H}\mathrm{P}^n = Sp(n+1)/Sp(n)Sp(1), \quad \mathbb{O}\mathrm{P}^2 = F_4/Spin(9);$$

2. Other homogeneous spheres [9, 43], i.e.,

$$S^{2n-1} = SU(n)/SU(n-1) = U(n)/U(n-1),$$
$$S^{4n-1} = Sp(n)/Sp(n-1) = Sp(n)U(1)/Sp(n-1)U(1) = Sp(n)Sp(1)/Sp(n-1)Sp(1),$$
$$S^6 = G_2/SU(3), \quad S^7 = Spin(7)/G_2, \quad S^{15} = Spin(9)/Spin(7),$$

   and the homogeneous complex projective space $\mathbb{C}\mathrm{P}^{2n-1} = Sp(n)/Sp(n-1)U(1)$;

3. two Berger spaces $Sp(2)/SU(2)$ and $SU(5)/Sp(2)U(1)$ [6];

4. three Wallach spaces $SU(3)/T^2$, $Sp(3)/Sp(1)^3$, $F_4/Spin(8)$ [52];

5. Aloff-Wallach spaces $SU(3)/S^1_{k,l}$, in which $k, l \in \mathbb{Z}$ satisfy $kl(k+l) \neq 0$ and $S^1_{k,l} = \{\mathrm{diag}(z^k, z^l, z^{-k-l}) | \forall z \in \mathbb{C}, |z| = 1\}$ [3].

See also [51], [54], [55] and [66] for some emendations and new observations. Since 1980's, a few new examples of closed manifolds with positive curvature have been found among biquotient and cohomogeneity one spaces in dimension 6, 7 and 13 [5, 18, 25, 26, 32].

4.2. Rank inequality and equivalent statement

Consider a connected Riemannian homogeneous space $M = G/H$, in which G is a closed connected subgroup of the connected isometric group of M, then H is a compact subgroup of G [10]. Bonnet-Myers Theorem [42] indicates the compactness of M. So G is also compact and the rank $\mathrm{rank}G$ is defined.

The rank inequality $\mathrm{rank}G \leq \mathrm{rank}H + 1$ was first observed by M. Berger, when he studied positively curved normal homogeneous Riemannian manifolds [6]. Since normal homogeneous Riemannian metric on G/H is induced by submersion from a bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on G ($\langle\cdot,\cdot\rangle_{\mathrm{bi}} = |\cdot|^2_{\mathrm{bi}}$ is also viewed as an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g} = \mathrm{Lie}(G)$), O'Neill formula [31] provides the curvature tensor of G/H, i.e.,

$$R(X, Y, Y, X) = \frac{1}{4}|[X, Y]|^2_{\mathrm{bi}} + \frac{3}{4}|[X, Y]_{\mathfrak{h}}|^2_{\mathrm{bi}}$$

for any $X, Y \in \mathfrak{m} = T_o(G/H)$, where the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is canonical. It is obvious to see that, when $\mathrm{rank}H \leq \mathrm{rank}G - 2$, we can find a linearly independent commuting pair X and Y in $\mathfrak{m}$. The tangent plane spanned by X and Y has zero sectional curvature. To summarize,

Lemma 4.1. *[6] A smooth coset space G/H with compact G and H admits a positively curved normal homogeneous Riemannian metric only when the rank inequality* $\mathrm{rank}G \leq \mathrm{rank}H + 1$ *is satisfied.*

When the positively curved homogeneous metric on G/H is not normal, the rank inequality was proved by N. Wallach in 1972 [52].

Theorem 4.2. *[52] Any positively curved Riemannian homogeneous space $M = G/H$ with compact G and H satisfies* $\mathrm{rank}G \leq \mathrm{rank}H + 1$.

In [52], N. Wallach also proved the following equivalent statement for Theorem 4.2.

Theorem 4.3. *[52] A connected Lie group admitting a positively curved left invariant Riemannian metric must be $SU(2)$ or $SO(3)$.*

To prove Theorem 4.3 from Theorem 4.2, we can just take the trivial $H = \{e\}$. To prove the inverse direction, we need a totally geodesic technique (i.e., fixed point set technique). Let T_H be a maximal torus in H. Its fixed point is nonempty, because it contains $o = eH$. The component $\mathrm{Fix}_o(T_H, G/H)$ containing o of the fixed point set of T_H in G/H is a totally geodesic submanifold which is locally isometric to a left-invariant metric on $C_G(T_H)$. The rank of $C_G(T_H)$, i.e., $\mathrm{rank}G - \mathrm{rank}H$, must be 0 or 1 by Theorem 4.3.

4.3. Proofs of the rank inequality

There are two proof of Theorem 4.3. The first is a topological proof proposed by N. Wallach [52], using Riemannian submersion and the following observation from M. Berger.

Lemma 4.4. *[7] Suppose M is an even dimensional closed Riemannian manifold with positive sectional curvature. Then any Killing field on M must have a zero.*

Proof. [Sketched topological proof of Theorem 4.3 in [52]] Suppose G is a compact connected Lie group endowed with a positively curved left invariant Riemannian metric. If $\mathrm{rank}G = 1$, then G is $SU(2)$ or $SO(3)$. Assume conversely $\mathrm{rank}G > 1$. Let T be a maximal torus in G, $T' \subset T$ a codimension-two sub-torus, and denote $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{t}' = \mathrm{Lie}(T')$. By Riemannian submersion, G/T' admits a positively curved homogeneous Riemannian metric induced from that on G. Any generic vector in $\mathfrak{t}\backslash\mathfrak{t}'$ induces a nowhere vanishing Killing vector field on G/T'. This is a contradiction to Lemma 4.4 because $\dim G/T'$ is even.

□

The computational method uses a technique from B. Wilking [55].

Proof. [Sketched computational proof of Theorem 4.3] Suppose the compact connected Lie group G is endowed with a left invariant Riemannian metric, corresponding to the inner product $\langle\cdot, L\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g} = \mathrm{Lie}(G)$, where L is a positive definite linear map with respect to the bi-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$. For any vector $X, Y \in \mathfrak{g} = T_eG$, the homogeneous curvature formula [8] provides

$$\begin{aligned} R(X,Y,Y,X) \quad = \quad & -\frac{3}{4}\langle[X,Y],[X,Y]\rangle + \frac{1}{2}\langle[[Y,X],Y],X\rangle + \frac{1}{2}\langle[[X,Y],X],Y\rangle \\ & +\langle U(X,Y),U(X,Y)\rangle - \langle U(X,X),U(Y,Y)\rangle, \end{aligned} \tag{4.1}$$

in which the bilinear function $U : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ is defined by $U(u,v) = \frac{1}{2}L^{-1}([u, Lv] + [v, Lu])$. When $\mathrm{rank}G > 1$, we can find a linearly independent commuting pair X and Z, such that X is an eigenvector for the smallest eigenvalue of L. So X and $Y = L^{-1}Z$ are also linearly independent. Calculation following (4.1) indicates $R(X,Y,Y,X) \leq 0$ (see [55][66] for the details). Notice that $R(X,Y,Y,X)$ is the nominator of the sectional curvature $K(e, X\wedge Y) = \frac{R(X,Y,Y,X)}{|X\wedge Y|^2}$, so $K(e, X\wedge Y) \leq 0$, i.e., the left invariant Riemannian metric on G is not positively curved.

□

4.4. Role of rank inequality in the classification

When $\mathrm{rank}G = \mathrm{rank}H$, the strong orthogonality provides a very convenient criterion for homogeneous positive curvature.

Lemma 4.5. *[52] Suppose the even dimensional coset space G/H with compact G and H admits a positively curved homogeneous Riemannian metric. Then it satisfies*
Condition (A)*: there do not exist a pair of linearly independent roots α and β of $\mathfrak{g}$ such that they are not roots of $\mathfrak{h}$ and $\alpha \pm \beta$ are not roots of $\mathfrak{g}$.*

The classification for the smooth compact coset space G/H satisfying Condition (A) results the Wallach's list in [52].

When $\text{rank}G = \text{rank}H + 1$, the case-by-case discussion for G/H in each of Category I-III (see Section 3.3) can be further simplified by totally geodesic subspace, block lemma and other techniques [55]. For almost all unwanted G/H, algebraic obstacles can be found from a linearly independent commuting pair from the $\mathfrak{m}$-factor in the canonical reductive decomposition for G/H, which spans a tangent plane with vanishing sectional curvature [8]. However, there exist a few exceptional cases, like $Sp(2)/U(1)$ in Category I, which admit homogeneous Riemannian metrics with positive curvature for all linearly independent commuting pairs in $\mathfrak{m}$ [66]. For these exceptional subcases, B. Wilking's technique (see the computational proof of Theorem 4.3) can provide the analytical obstacle to homogeneous positive curvature [55, 66].

5. Positive flag curvature in homogeneous Finsler geometry

In Finsler geometry, the positive curvature or positively curved property is referred to metrics or manifolds with positive flag curvature. Closed Finsler manifolds with positive curvature share many geometric and topological properties with those in Riemannian geometry. For example, Bonnet-Myers Theorem [42] and Synge Theorem [50] in Riemannian geometry can be immediately generalized [11], and we can still use the submersion and totally geodesic techniques [2, 17] in Finsler geometry. On the other hand, the variational method [49] only works partially, because of the base vector issue in Finsler geometry, so some other important theories, like the triangular comparison [16] and Frankel Theorem [29], are no longer valid in Finsler geometry.

5.1. Rank inequality for homogeneous positive flag curvature

To systematically study the positive curvature problem in Finsler geometry, classifying the positively curved homogeneous Finsler spaces is the start point. Due to the complexity of calculation in Finsler geometry, it has not been touched until the 2010's. A remarkable progress is the following theorem of S. Deng and Z. Hu [21], which generalizes Theorem 4.3,

Theorem 5.1. *[21] Let G be a connected compact Lie group which admits a left invariant Finsler metric with positive flag curvature, then $G = SU(2)$ or $SO(3)$.*

By the same totally geodesic technique as in Riemannian geometry, Theorem 5.1 is equivalent to the following rank inequality for positively curved homogeneous Finsler spaces [65].

Theorem 5.2. *[65] Let $(G/H, F)$ be a positively curved homogeneous Finsler space with compact G and H. Then it satisfies the rank inequality* $\text{rank}G \leq \text{rank}H + 1$.

The proof of Theorem 5.1 in [21] is topological, which is similar to that for Theorem 4.3 in [52]. In [35], L. Huang proposed a computational proof of Theorem 5.1 using his homogeneous curvature formula [34], i.e.,

Theorem 5.3. *[34] Let $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then for any linearly independent pair $y, v \in \mathfrak{m} = T_e(G/H)$, the flag curvature for the triple $(o = eH, y, \mathbf{P} = \text{span}\{y, v\})$ is*

$$K(o, y, \mathbf{P}) = \frac{g_y([[v, y]_\mathfrak{h}, v], y) + g_y(\tilde{R}(v), v)}{g_y(y, y)g_y(v, v) - g_y(y, v)^2},$$

in which

$$\tilde{R}(v) = (D_\eta N)(y, v) - N(y, N(y, v)) + N(y, [y, v]_\mathfrak{m}) - [y, N(y, v)]_\mathfrak{m}. \tag{5.1}$$

Here $g_y(\cdot, \cdot)$ denote the fundamental tensor for the $\text{Ad}(H)$-invariant Minkowski norm F determines in $\mathfrak{m}$, $\eta : \mathfrak{m}\backslash\{0\} \to$ is the spray vector field defined by

$$g_y(\eta(y), v) = g_y(y, [v, y]_\mathfrak{m}),$$

$N : \mathfrak{m}\backslash\{0\} \times \mathfrak{m} \to \mathfrak{m}$ is the connection operator defined by

$$2g_y(N(y, v), u) = g_y([u, v]_\mathfrak{m}, y) + g_y([u, y]_\mathfrak{m}, v) + g_y([v, y]_\mathfrak{m}, u) - 2C_y(u, v, \eta(y)),$$

and $(D_\eta N)(y, v)$ is the directional derivative of $N(\cdot, v)$ in the direction of $\eta(y)$ at y.

Proof. [Sketched computational proof of Theorem 5.1 in [35]] B. Wilking's technique is applied as following. Suppose F is a left invariant Finsler metric on the compact Lie group G with $\text{rank}G > 1$. For simplicity, we also use F to denote the Minkowski norm it defines in $T_eG = \mathfrak{g}$. We fix a bi-invariant inner product on $\mathfrak{g}$, i.e., $|\cdot|^2_{\mathrm{bi}} = \langle\cdot,\cdot\rangle_{\mathrm{bi}}$. The flag triple $(e, y, \mathbf{P} = \mathrm{span}\{y, v\})$ is determined as following. Firstly, the nonzero $y \in \mathfrak{g}$ is chosen where the function $f(\cdot) = \frac{F(\cdot)}{|\cdot|_{\mathrm{bi}}}$ achieves its minimum. The fundamental tensor $g_y(\cdot,\cdot)$ at y determines a positive definite linear operator L for $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$, i.e., $g_y(\cdot,\cdot) = \langle\cdot, L\cdot\rangle_{\mathrm{bi}}$, and y is an eigenvector for the minimal eigenvalue of L. Nextly, we use the assumption $\mathrm{rank}G > 1$ to find $v' \in \mathfrak{g}\backslash\mathbb{R}y$ commuting with y. Then y and $v = L^{-1}v'$ are linearly independent. Finally, we use the L. Huang's homogeneous flag curvature formula, i.e., Theorem 5.3, to calculate the flag curvature $K(e, y, \mathbf{P})$. Notice that the choice of y implies $\eta(y) = 0$, so the calculation can be simplified and is similar to that in the computational proof of Theorem 4.3, where we obtain a non-positive curvature.

□

5.2. Classification of positively curved homogeneous Finsler spaces

To study the classification for homogeneous positive curvature in Finsler geometry, we need to check case by case each G/H searching for obstacles to positive curvature, where the algebraic setup in Section 3 can be applied because of the rank inequality in Theorem 5.2. The homogeneous flag curvature formula provides the criteria. Since the formula (5.1) is still quite complicated, the following simplified version [65] (which can be deduced from Theorem 5.3) is more useful.

Theorem 5.4. *[65] Let $(G/H, F)$ be a connected homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then for any linearly independent commuting pair $u, v \in \mathfrak{m} = T_o(G/H)$ satisfying $g_u([u,\mathfrak{m}]_\mathfrak{m}, u) = 0$, we have*

$$K(o, u, u \wedge v) = \frac{g_u(U(u,v), U(u,v))}{g_u(u,u)g_u(v,v) - g_u(u,v)^2},$$

where $U(u,v) \in \mathfrak{m}$ is determined by

$$g_u(U(u,v), w) = \frac{1}{2}(g_u([w,u]_\mathfrak{m}, v) + g_u([w,v]_\mathfrak{m}, u)), \quad \forall w \in \mathfrak{m}.$$

In a series of works, we partially generalized the classification for positively curved homogeneous spaces from Riemannian geometry to Finsler geometry:

1. when the positively curved $(G/H, F)$ is normal homogeneous or generalized normal homogeneous (i.e., δ-homogeneous [14, 68]), the classifications are complete, which coincide with M. Berger's list [6], i.e., S^n, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, $\mathbb{O}\mathrm{P}^2$, two Berger spaces $Sp(2)/SU(2)$ and $SU(5)/Sp(2)U(1)$, and the addon $SU(3) \times SO(3)/U(2) = SU(3)/S^1_{1,1}$ by B. Wilking [62, 68];

2. when the positively curved $(G/H, F)$ is even dimensional, the classification list is complete, which coincide with N. Wallach's list [52], i.e., S^{2n}, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, $\mathbb{O}\mathrm{P}^2$, and the three Wallach spaces $SU(3)/T^2$, $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$ [65];

3. when G/H is odd dimensional and F is positively curved and reversible, then there are two possibilities [63, 67]. Either G/H admits positively curved homogeneous Riemannian metric, i.e., it belongs to L. Bergery's list [8], or it belongs to one of following five candidates,

$$\begin{aligned}
&Sp(2)/\mathrm{diag}(z, z^3) \quad \text{with} \quad z \in \mathbb{C},\\
&Sp(2)/\mathrm{diag}(z, z) \quad \text{with} \quad z \in \mathbb{C},\\
&Sp(3)/\mathrm{diag}(z, z, q) \quad \text{with} \quad z \in \mathbb{C}, q \in \mathbb{H},\\
&SU(4)/\mathrm{diag}(zA, z, \bar{z}^3) \quad \text{with} \quad A \in SU(2), z \in \mathbb{C},\\
&G_2/SU(2) \quad \text{with normal } SU(2) \text{ in } SO(4) \subset G_2 \text{ corresponding to a long root;}
\end{aligned}$$

4. If we further assume that $(G/H, F)$ in (3) is of (α, β)-type, then only two undetermined candidates remains [67]:

$$\begin{aligned}
&Sp(2)/\mathrm{diag}(z, z) \quad \text{with} \quad z \in \mathbb{C},\\
&Sp(3)/\mathrm{diag}(z, z, q) \quad \text{with} \quad z \in \mathbb{C}, q \in \mathbb{H};
\end{aligned}$$

5. If we assume $(G/H, F)$ is a positively curved homogeneous (α, β)-space with vanishing (or equivalently, isotropic [60]) S-curvature [47], then G/H can be completely classified [61], i.e., it is one of the odd dimensional homogeneous spheres $S^{2n-1} = SU(n)/SU(n-1) = U(n)/U(n-1)$ and $S^{4n-1} = Sp(n)/Sp(n-1) = Sp(n)U(1)/Sp(n-1)U(1)$, or one of Aloff-Wallach spaces;

6. If we further assume $(G/H, F)$ in (4) is Randers, then the Killing navigation process [27, 30, 37] can be applied to determine all positively curved metrics [36].

6. Homogeneous Finsler manifold satisfying $K \geq 0$ and the (FP) condition

6.1. (FP) condition

Though flag curvature in Finsler geometry is a perfect generalization for sectional curvature in Riemannian geometry, it is much more local because it depends not only on the flag but also the flagpole. This observation inspires a new positive curvature property in Finsler geometry.

Definition 6.1. *[64] We say the Finsler manifold (M, F) is flagwise positively curved or it satisfies the (FP) condition, if for any $x \in M$ and any tangent plane $\mathbf{P} \subset T_xM$, we can find a nonzero $y \in \mathbf{P}$, such that the flag curvature $K(x, y, \mathbf{P}) > 0$.*

It is easy to see that in Riemannian geometry, a metric is flagwise positively curved iff it is positively curved.

The (FP) condition in Finsler geometry is much weaker than the positive flag curvature property and metrics satisfying the (FP) condition are not hard to find. For example, for any smooth compact coset space G/H with a finite fundamental group, G/H and $G/H \times S^1$ admit (generally non-homogeneous) flagwise positively curved Finsler metrics; any Lie group G with $\dim G > 1$ admits left invariant metrics satisfying the (FP) condition if its Lie algebra $\mathfrak{g}$ is compact and $\mathfrak{c}(\mathfrak{g}) < 2$ [56]. It is believed that we can get a flagwise positively curved Finsler metric by generically perturbing any non-negatively curved Finsler metric.

6.2. Homogeneous Finsler spaces satisfying $K \geq 0$ and the (FP) condition

The combination of the (FP) condition and the non-negatively curved condition is believed to be an interesting and reasonable approximation for the positive flag curvature property [64]. We can use the Killing navigation process [27] to find many compact coset spaces which admit homogeneous Finsler metrics satisfying $K \geq 0$ and the (FP) condition, but do not admit positively curved homogeneous Finsler metrics. Here are some explicit examples [64],

$$\begin{aligned}
&SU(p+q)/SU(p)SU(q), \quad \text{with } p > q \geq 2 \text{ or } p = q > 3,\\
&Sp(n)/SU(n), \quad \text{with } n > 4,\\
&SO(2n)/SU(n), \quad \text{with } n = 5 \text{ or } n > 6,\\
&E_6/SO(10) \quad \text{and} \quad E_7/E_6.
\end{aligned}$$

6.3. Compactness and rank inequality conjecture

It is intriguing to explore what geometric or topological properties of positively curved homogeneous Finsler metrics are preserved by homogeneous Finsler spaces satisfying $K \geq 0$ and the (FP) condition. So we proposed the following *compactness and rank inequality conjecture* in [64].

Conjecture 6.2. *[64] Suppose (M, F) is a homogeneous Finsler manifold satisfying $K \geq 0$ and the (FP) condition, then M is compact. Further more, if we present $M = G/H$ with compact G and H, then we have* $\mathrm{rank} G \leq \mathrm{rank} H + 1$.

Conjecture 6.2 can be viewed as the generalizations for the Bonnet-Myers Theorem and the rank inequality for positively curved homogeneous Finsler spaces.

Until now, Conjecture 6.2 has been proved for some special cases.

Recall that a (generalized) normal homogeneous Finsler metric F on G/H is induced by submersion from a bi-invariant metric $\bar{F}$ on G. Notice that though we require F to be smooth, $\bar{F}$ could be singular. Normal homogeneous and generalized normal homogeneous Finsler space are always non-negatively curved and they can be presented as G/H with compact $\mathfrak{g} = \mathrm{Lie}(G)$ [62, 68]. Conjecture 6.2 has been proved for these cases.

Theorem 6.3. *[68] Let $(G/H, F)$ be a normal or generalized normal homogeneous Finsler manifold satisfying the (FP) condition, then G/H is compact with compact $\mathfrak{g} = \mathrm{Lie}(G)$, and the rank inequality* $\mathrm{rank} G \leq \mathrm{rank} H + 1$ *is satisfied.*

Theorem 6.3 has the following immediate consequence.

Corollary 6.4. *[68] The following statements for the smooth compact coset space G/H with compact G and H are equivalent:*

1. *G/H admits positively curved normal homogeneous Riemannian metric;*
2. *G/H admits positively curved (generalized) normal homogeneous Finsler metric;*

3. G/H admits positively curved (generalized) normal homogeneous Finsler metric satisfying the (FP) condition.

It provides a complete classification for normal and generalized normal homogeneous Finsler spaces with an even dimension which satisfy the (FP) condition, i.e., homogeneous S^{2n}, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ and $\mathbb{O}\mathrm{P}^2$.

Normal homogeneity and generalized normal homogeneity is special cases of the geodesic orbit property [40][69], i.e., any geodesic is the orbit of a one-parametric isometric subgroup. In [59], Conjecture 6.2 is proved for geodesic orbit Finsler spaces.

Theorem 6.5. *[59] Let (M, F) be a geodesic orbit Finsler manifold satisfying $K \geq 0$ and the (FP) condition, then M is compact. If we present $M = G/H$ with compact G and H, then the rank inequality* $\mathrm{rank}G \leq \mathrm{rank}H + 1$ *is satisfied.*

Following after Theorem 6.5, we can get an incomplete classification for geodesic orbit Finsler spaces, i.e.,

Corollary 6.6. *[59] If the even dimensional smooth compact coset space G/H admits a geodesic orbit Finsler metric satisfying $K \geq 0$ and the (FP) condition, then it admits a positively curved metric, i.e., G/H must be one of homogeneous S^{2n}, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ and $\mathbb{O}\mathrm{P}^2$, or one of the three Wallach spaces $SU(3)/T^2$, $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$.*

The classification in Corollary 6.6 is incomplete because we can not determine if the three Wallach spaces admits geodesic orbit Finsler metrics satisfying $K \geq 0$ and the (FP) condition. In Riemannian geometry, the geodesic orbit metrics on the three Wallach spaces are normal homogeneous [4], which are not positively curved [4]. But in Finsler geometry, there exists many other geodesic orbit Finsler metrics which are not normal homogeneous or generalized normal homogeneous [72].

7. Existence of two closed geodesic orbits

7.1. Orbit number of prime closed geodesics

Closed geodesic is another important topic in Finsler geometry [39]. The project of estimating the number of closed geodesics on a closed Finsler manifold attracts many attentions. A. Katok found Finsler spheres with only finitely many closed geodesics [38]. Katok spheres are examples of Randers spheres with constant flag curvature [15]. Recently, A. Katok's observation have been generalized to describe geodesic behaviors on other Finsler spheres with constant curvature [12, 57].

When we count the number of closed geodesics, repeating rotations are ignored, i.e., we only count those prime ones. However, a reversible closed geodesic, i.e., both $c(t)$ and $c(-t)$ are closed geodesics, possibly with nonconstant speed, it is counted twice. The existence of the first prime closed geodesic on a closed Finsler manifold has been well known since 1960's [28]. The existence of the second closed geodesic on a closed Finsler manifold is much harder, which has been extensively explored in recent years [13, 22, 46]. More accurate estimates are conjectured for Finsler metrics on compact rank-one symmetric spaces S^n, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ and $\mathbb{O}\mathrm{P}^2$ based on Katok spheres and other Randers metric induced from the standard Riemannian symmetric metrics by Killing navigation [70]. For example, D. Anosov conjectured the number of closed geodesics on a Finsler sphere (S^n, F) is at least $2[\frac{n+1}{2}]$ [1]. Since closed geodesics on a closed Finsler manifold (M, F) can be viewed as the critical points for the energy functional defined on the free loop space ΛM, Morse theory and equivariant topology are the usual tools studying these closed geodesic problems [53].

Now we assume the closed Finsler manifold (M, F) admits a nontrivial continuous isometric action, i.e., its connected isometry group $G = I_0(M, F)$ has a positive dimension. Then we observe that the Lie method plays a dominant role and closed geodesics can be easily found. For example,

Lemma 7.1. *[58] Let (M, F) be a closed Finsler manifold with* $\dim I(M, F) > 0$*. Then there exist two different closed geodesics on (M, F).*

Proof. Denote $G = I_0(M, F)$ the connected isometry group, and $\mathfrak{g} = \mathrm{Lie}(G)$. Since G is compact and has a positive dimension, we can find a nonzero vector $v \in \mathfrak{g}$ which generates an S^1-subgroup. Let V be the Killing field induced by v and $f_1(\cdot) = F(V(\cdot))$. At any critical point of $f_1(\cdot)$ where $f_1(\cdot)$ is positive (the maximum point of $f_1(\cdot)$ for example), V generates a closed geodesic (see Lemma 3.1 in [23]). Consider $f_2(\cdot) = F(-V(\cdot))$ instead, similar argument proves $-V$ generates another closed geodesic.

□

Meanwhile, it is easy to see closed geodesics appear in orbits. The free loop space ΛM of maps $c(t):\mathbb{R}/\mathbb{Z}\to M$ admits the natural action $\hat{G}=G\times S^1$, i.e. $\forall g\in G=I_0(M,F), t'\in S^1=\mathbb{R}/\mathbb{Z}$, $(g\cdot c)(t)=g(c(t+t'))$. This action preserves the subset of closed geodesics in ΛM. When $\dim G>0$, to avoid a trivial infinity for the number of closed geodesics, it is more reasonable to estimate the number of $\hat{G}$-orbits of prime closed geodesics on (M,F) [57].

Inspired by [13, 22, 46] and other literature, we proposed the project to explore the existence of two orbits of prime closed geodesics on a closed Finsler manifold. However, this is not always true. When (M,F) is isometric to one of the compact rank-one Riemannian symmetric spaces, S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ and $\mathbb{O}\mathrm{P}^2$, it has only one orbit of prime closed geodesics. Until now, there are no other known examples. So we conjecture

Conjecture 7.2. *[58] Suppose the closed connected Finsler manifold (M,F) satisfies that $\dim I(M,F)>0$ and it has only one orbit of prime closed geodesics, then it is isometric to a Riemannian symmetric S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ or $\mathbb{O}\mathrm{P}^2$.*

7.2. Rank inequality for only one orbit of prime closed geodesics

To study Conjecture 7.2, we assume conversely that the connected closed Finsler manifold (M,F) has only one $\hat{G}$-orbit of prime closed geodesics. Then as shown in Section 7.1, we can use a nonzero vector in $\mathfrak{g}=\mathrm{Lie}(G)$ to generate two different closed geodesics, which are homogeneous, i.e., orbits of one-parameter subgroups in G. The following properties of (M,F) can be derived from this observation.

Lemma 7.3. *[58] Suppose (M,F) is a closed Finsler manifold with $\dim I(M,F)>0$ and has only one orbit of prime closed geodesics. Then $G=I_0(M,F)$ is semi-simple, all the closed geodesics on (M,F) are homogeneous, i.e., orbits of one-parameter subgroups in G, and the union of all closed geodesics is a G-orbit in M.*

Amazingly, the rank inequality appears here.

Theorem 7.4. *[58] Suppose (M,F) is a closed Finsler manifold with $\dim I(M,F)>0$ and has only one orbit of prime closed geodesics. Let $N=G\cdot x=G/H$ with $G=I_0(M,F)$ be the orbit containing all closed geodesics. Then we have the rank inequality* $\mathrm{rank}G\leq\mathrm{rank}H+1$.

Proof. [Sketched proof of Theorem 7.4][58] To prove the rank inequality, we denote $c(t):\mathbb{R}/\mathrm{Z}\to M$ a closed geodesic passing x, H the isotropy subgroup at x, and two compact subgroups of G,

$$\begin{aligned} H_1 &= \{g\in G|\exists t_0\in\mathbb{R}/\mathbb{Z} \text{ with } g(c(t))=c(t+t_0),\forall t\},\\ H_2 &= \{g\in G|g(c(t))=c(t),\forall t\}.\end{aligned}$$

Obviously $H_2=H_1\cap H$ is a normal subgroup of H_1 and $H_1/H_2=S^1$. Denote $\mathfrak{h}_i=\mathrm{Lie}(H_i)$.

By the compactness of G, $\mathcal{U}=\bigcup_{g\in G}\mathrm{Ad}(g)\mathfrak{h}_1$ is a closed subset in $\mathfrak{g}$. Further more, We must have $\mathcal{U}=\mathfrak{g}$, otherwise we can find a nonzero vector $v\in\mathfrak{g}\backslash\mathcal{U}$ which generates an S^1-subgroup. The Killing vector field V on (M,F) induced by v can generate some nonconstant closed geodesic passing $g\cdot x$ for some $g\in G$. Then v is contained in $\mathrm{Ad}(g^{-1})\mathfrak{h}_1$, which contradicts $v\in\mathfrak{g}\backslash\mathcal{U}=\mathfrak{g}\backslash\bigcup_{g\in G}\mathrm{Ad}(g)\mathfrak{h}_1$.

To summarize, $\mathfrak{h}_1$ contains a generic vector in $\mathfrak{g}$ which generates a dense one-parameter subgroup in a maximal torus of G. So H_1 must contain a maximal torus of G, i.e., $\mathrm{rank}H_1=\mathrm{rank}G$. Then we have

$$\mathrm{rank}H\geq\mathrm{rank}H_2=\mathrm{rank}H_1-1=\mathrm{rank}G-1,$$

which ends the proof.

□

7.3. Compact homogeneous Finsler spaces with only one closed geodesic orbit

Now we assume $(G/H,F)$ is a compact homogeneous Finsler manifold with $G=I_0(G/H,F)$, which only has one orbit of prime closed geodesics. Theorem 7.4 provides the rank inequality $\mathrm{rank}G\leq\mathrm{rank}H+1$. Then we can use the algebraic setup in Section 3 to build a case-by-case discussion and prove Conjecture 7.2 in the following special cases.

Theorem 7.5. *[58] Assume (M,F) is a compact connected homogeneous Finsler space with only one orbit of prime closed geodesics. If $\dim M$ is odd, we further assume F is reversible. Then (M,F) must be a Riemannian symmetric S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$ or $\mathbb{O}\mathrm{P}^2$.*

Notice that Theorem 7.5 is not a trivial fact when F is reversible or Riemannian. Though any reversible closed geodesic provides two different prime closed geodesics (corresponding to its two directions), it is possible that both are contained in the same closed geodesic orbit, as any compact rank-one symmetric space shows.

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