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Original Article

On Finsler warped product metrics with vanishing E-curvature

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ABSTRACT: In this paper, we study Finsler warped product metric recently introduced by P. Marcal and Z. Shen and find characteristics differential equations for this metric to have vanishing E-curvature. We also prove that if this warped product Finsler metric is projectively flat, then it becomes a Riemannian metric.

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(Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration)

1. Introduction

R. L. Bishop and B. O'Neill introduced the warped product Riemannian manifold in 1964 [3], to construct a class of complete Riemannian manifolds of negative curvature. Warped product Riemannian manifolds are the most natural and fruitful generalization of Riemannian products of two manifolds. The notion of warped products play very important roles not only in geometry but also in mathematical physics, especially in general relativity. In fact, many basic solutions of the Einstein field equations, including the Schwarzschild solution and the Robertson-Walker models, are warped product manifolds. The famous John Nash's embedding theorem published in 1956 implies that every warped product Riemannian manifold can be realized as a warped product submanifold in a suitable Euclidean space [13, 14].

A Finsler metric on a smooth manifold is a smoothly varying family of Minkowski norms, one on each tangent space, rather than a family of inner products one on each tangent space, as in the case of Riemannian metrics. It turns out that every Finsler metric induces an inner product in each direction of a tangent space at each point of the manifold. However, in sharp contrast to the Riemannian case, these Finsler-inner products do not only depend on where we are, but also in which direction we are looking.

G.S. Asanov [1, 2], generalized the Schwarzschild metric in the Finslerian setting and obtained some models of relativity theory described through the warped product of Finsler metrics. In [17], Z. Shen used a construction

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of warped Riemannian metrics at the vertical bundle, and obtained a Finslerian warped product metric. Further, Kozma–Peter–Varga used the Finsler fundamental functions to define their warped product [9]. Then they studied the relationships between the Cartan connection of the doubly warped product manifold and its components. In the last decade some significant progress has been made in the study of Finsler warped product metrics [5, 9, 10]. M. M. Rezaii and Y. Alipour-Fakhari studied the projectively related warped product Finsler metrics and find a non-linear connection on the product of two Finsler manifolds [16].

It has been observed in [5] that spherically symmetric Finsler metrics have warped product structure. However, there are a lot of Finsler warped product metrics (e.g. general (α, β)-metrics) that are not spherically symmetric and several authors studied those metrics [20, 21, 4, 22]. In [12], the authors obtain the differential equation that characterizes the spherically symmetric Finsler metrics with vanishing Douglas curvature. Furthermore, they obtain all the spherically symmetric Douglas metrics by solving this equation. E. Peyghan and A. Tayebi studied the horizontal and vertical warped product Finsler manifolds and prove that every C-reducible or proper Berwaldian doubly warped product Finsler manifold is Riemannian [15]. There are several non-Riemannian quantities in the Finsler literature, for instance, Cartan torsion, S-curvature, E-curvature, Ξ-curvature, H-curvature etc. These quantities become zero for a Riemannian metric. In [12], the authors obtain the volume form of warped product of Finsler metrics defind in [5] and prove the notions of isotropic E-curvature and isotropic S-curvature are equivalent to each other [8].

Let us consider the product manifold $M = I \times \bar{M}$, where I is an interval of $\mathbb{R}$ and $(\bar{M}, \bar{\alpha})$ is an $(n-1)$ dimensional Riemannian manifold. Let $\{\theta^a\}_{a=2}^n$ be a local coordinate system on $\bar{M}$. Then $\{u^i\}_{i=1}^n$ gives us a local coordinate on M by setting $u^1 = r$ and $u^a = \theta^a$. The indices $i, j, k, \ldots$ are ranging from 1 to n while $a, b, c, \ldots$ are ranging from 2 to n. A vector v on M can be written as $v = v^i \partial/\partial u^i$ and its projection on $\bar{M}$ is denoted by $\bar{v} = v^a \partial/\partial u^a = v^a \partial/\partial \theta^a$. A warped product Finsler metric can be written in the form

$$F = \bar{\alpha}\sqrt{w(s, r)},$$

where w is a suitable function defined on an open subset of $\mathbb{R}^2$ and $s = v^1/\bar{\alpha}$. It can be rewritten as [5]

$$F = \bar{\alpha}\phi(s, r), \qquad \text{where} \quad \phi(s, r) = \sqrt{w(s, r)}.$$

In this paper, we consider the warped product Finsler metrics which are similar to the warped product metric defined in [11] but with a different warping which was introduced by P. Marcal and Z. Shen in [11]. Let $\mathbb{R}$ and $\mathbb{R}^n$ are Riemannian manifolds with the Euclidean metrics dt^2 and α^2 respectively. Then, we can define a Finsler metric on $\mathbb{R} \times \mathbb{R}^n$ by

$$F(z) = \alpha\sqrt{\phi(z, \rho)}, \tag{1.1}$$

where $z = \frac{dt}{\alpha}$ and $\rho = |\bar{x}|$, for $\bar{x} \in \mathbb{R}^n$, ϕ is a smooth real valued function on $\mathbb{R}^2$. In this paper we find characterization differential equations for the metric defined in (1.1) to be of vanishing E- curvature. We also prove that this Finsler warped product metric is projecively flat if and only of it is a Riemannian metric. More precisely, we obtain the following results:

Theorem 1.1. *Any warped product Finsler metric $F(z) = \alpha\sqrt{\phi(z, \rho)}$ defind in (1.1) has vanishing E-curvature if and only if the following condition holds:*

$$\Psi_z = 0,$$

where Ψ is defined in (3.3).

Further, we also prove the following result:

Theorem 1.2. *Any warped product Finsler metric $F(z) = \alpha\sqrt{\phi(z, \rho)}$ defind in (1.1) is projectively flat if and only if it is a Riemannian metric.*

2. Preliminaries

Definition 2.1. *A Finsler metric on M is a function $F : TM \to [0, \infty)$ satisfying the following conditions:*
(i) F is smooth on TM_0,
(ii) F is a positively 1-homogeneous on the fibers of tangent bundle TM,
(iii) The Hessian of $\frac{1}{2}F^2$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.
The pair (M, F) is called a Finsler space and g_{ij} is called the fundamental metric tensor.

Let us consider, the Finsler manifold $M = \mathbb{R} \times \mathbb{R}^n$ with the coordinates on TM as $(x^0, \bar{x}), where\ \bar{x} = (x^1, x^2, ..., x^n)$ and $(y^0, \bar{y}),\ \bar{y} = (y^1, y^2, ..., y^n)$ and the Finsler metric on M is given by

$$F(z) = \alpha\sqrt{\phi(z, \rho)}, \tag{2.1}$$

where, $\alpha = |\bar{y}|$, $z = \frac{y^0}{|\bar{y}|}$ $\rho = |\bar{x}|$. The Fundamental metric tensor of the Finsler metric defined in (2.1) is given by [11]

$$\begin{pmatrix} g_{11} & g_{1j} \\ g_{i1} & g_{ij} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\phi_{zz} & \frac{1}{2}\Omega_z\frac{\delta_{ij}y^i}{\alpha} \\ \frac{1}{2}\Omega_z\frac{\delta_{ij}y^j}{\alpha} & \frac{1}{2}\Omega\delta_{ij} - \frac{1}{2}z\Omega_z\frac{\delta_{il}y^l\delta_{mj}y^m}{\alpha^2} \end{pmatrix},$$

where

$$\Omega = 2\phi - z\phi_z$$

and the determinant of the above matrix is given by

$$\frac{1}{2^{n+1}}\Omega^{n-1}\Delta,$$

here

$$\Delta = \phi_{zz}(\Omega - z\Omega_z) - \Omega_z^2 = 2\phi\phi_{zz} - \phi_z^2.$$

Therefore, the contravariant metric tensor of the Finsler metric defined in (2.1) is given by

$$\begin{pmatrix} g^{11} & g^{1j} \\ g^{i1} & g^{ij} \end{pmatrix} = \begin{pmatrix} \frac{2}{\Lambda}(\Omega - z\Omega_z) & \frac{2}{\Delta}\Omega_z\frac{y^j}{\alpha} \\ \frac{2}{\Delta}\Omega_z\frac{y^i}{\alpha} & \frac{2}{\Omega}\delta^{ij} + \frac{2\phi_z\Omega_z}{\Omega\Delta}\frac{y^iy^j}{\alpha^2} \end{pmatrix}.$$

On a Finsler manifold (M, F), the geodesics are characterized by

$$\frac{d^2x^a}{dt^2} + G^a(x, \frac{dx}{dt}) = 0 \quad \forall\ a = 0, 1, ..., n,$$

where G^a are called spray coefficients of F, and defined by

$$G^a = \frac{1}{4}g^{ab}\left\{[F^2]_{y^cy^b}y^c - [F^2]_{x^b}\right\} \quad \forall\ a, b, c = 0, 1, ..., n.$$

Proposition 2.2. *[11] The spray coefficients of warped product Finsler metrics F are given by*

$$G^0 = (U + Vz)P\alpha, \quad G^i = (V + W)y^iP - Wx^i\alpha^2, \quad \forall\ i = 1, ..., n, \tag{2.2}$$

where,

$$U = \frac{1}{2\rho\Delta}(2\phi\phi_{z\rho} - \phi_z\phi_\rho), \quad V = \frac{1}{2\rho\Delta}(\phi_\rho\phi_{zz} - \phi_z\phi_{z\rho}), \quad W = \frac{1}{2\rho\Omega}\phi_\rho, \quad P = \sum_{k=1}^{n}x^ky^k$$

3. Warped product Finsler metrics with vanishing E-curvature

Definition 3.1. The E-curvature of a Finsler metric F is defined as

$$E_{ij} := \frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right),$$

where G^i are the spray coefficients of the Finsler metric F.
The Finsler metric F is said to be of vanishing E-curvature if $E_{ij} = 0$.

Before proving Theorem 1.1 we need the following results. By some simple calculations we get,

$$\alpha_{y^i} = \frac{y^i}{\alpha}, \qquad z_{y^0} = \frac{1}{\alpha}, \qquad z_{y^0}y^0 = z, \tag{3.1}$$

$$z_{y^i} = -\frac{zy^i}{\alpha^2}, \quad z_{y^i}y^i = 0, \tag{3.2}$$

$$z_{y^0y^j} = -\frac{y^j}{\alpha^3}, \qquad z_{y^iy^j} = \frac{3zy^iy^j - z\alpha^2\delta^i_j}{\alpha^4},$$

$$\alpha_{y^iy^j} = \frac{\alpha^2\delta^i_j - y^iy^j}{\alpha^3}\ \forall\ i = 1, ..., n.$$

Now in the next proposition, we obtain the E-curvature of the warped product Finsler metric F.

Proposition 3.2. *Let $F = \bar{\alpha}\phi(r,s)$ be a warped product Finsler metric defind in (1.1). Then E-curvature of F is given by,*

$$E_{00} = \frac{1}{2}\Psi_{zz}\frac{P}{\alpha^2}$$

$$E_{0i} = E_{i0} = -\frac{1}{2}\left[z\Psi_{zz} + \Psi_z\right]\frac{Py^i}{\alpha^3} + \frac{1}{2}\Psi_z\frac{x^i}{\alpha},$$

$$E_{ij} = \frac{1}{2}\left[z^2\Psi_{zz} + 3z\Psi_z\right]\frac{Py^iy^j}{\alpha^4} - \frac{1}{2}z\Psi_z\frac{\left(x^iy^j + x^jy^i\right)}{\alpha^2} - \frac{1}{2}z\Psi_z\frac{P}{\alpha^2}\delta_{ij},$$

where

$$\Psi = U_z + zV_z + V + zW_z - 2W + (n+1)(V+W). \tag{3.3}$$

Proof. Differentiating G^0 with respect to y^0 we have,

$$\frac{\partial G^0}{\partial y^0} = (U_z + V + zV_z)P \tag{3.4}$$

And differentiating G^i with respect to y^k we get,

$$\frac{\partial G^i}{\partial y^k} = (V_z + W_z)z_{y^k}Py^i + (V+W)x^ky^i + (V+W)P\delta_{ik} + zW_zx^iy^k - 2Wx^iy^k$$

Putting $i = k$, without taking summation over k and then using equation (3.1) and equation (3.2), we get

$$\frac{\partial}{\partial y^k}(G^k) = (V+W)x^ky^k + (V+W)P + (zW_z - 2W)x^ky^k \tag{3.5}$$

From (3.4) and (3.5), we therefore have

$$\sum_{a=0}^{n}\frac{\partial G^a}{\partial y^a} = \frac{\partial G^0}{\partial y^0} + \sum_{k=1}^{n}\frac{\partial G^k}{\partial y^k} = [U_z + zV_z + V + zW_z - 2W + (n+1)(V+W)]P \tag{3.6}$$

Let us assume $\Psi = U_z + zV_z + V + zW_z - 2W + (n+1)(V+W)$. Then $\sum_{a=0}^{n}\frac{\partial G^a}{\partial y^a} = \Psi P$.
Differentiating (3.6) with respect to y^0 twice gives,

$$E_{00} = \frac{1}{2}\Psi_{zz}\frac{P}{\alpha^2} \tag{3.7}$$

Differentiating (3.6) first with respect to y^0 and then with respect to y^i gives,

$$E_{0i} = E_{i0} = -\frac{1}{2}\left[z\Psi_{zz} + \Psi_z\right]\frac{Py^i}{\alpha^3} + \frac{1}{2}\Psi_z\frac{x^i}{\alpha} \tag{3.8}$$

Differentiating (3.6) with respect to y^i, and then with respect to y^j gives,

$$E_{ij} = \frac{1}{2}\left[z^2\Psi_{zz} + 3z\Psi_z\right]\frac{Py^iy^j}{\alpha^4} - \frac{1}{2}z\Psi_z\frac{\left(x^iy^j + x^jy^i\right)}{\alpha^2} - \frac{1}{2}z\Psi_z\frac{P}{\alpha^2}\delta_{ij} \tag{3.9}$$

□

Proof of Theorem 1.1
The Finsler metric F has vanishing E-curvature if and only if $E_{ij} = 0$. Therefore, equations (3.7), (3.8) and (3.9) gives

$$\Psi_{zz} = 0 \tag{3.10}$$

$$z\Psi_{zz} + \Psi_z = 0 \tag{3.11}$$

$$\Psi_z = 0 \tag{3.12}$$

$$z^2\Psi_{zz} + 3z\Psi_z = 0 \tag{3.13}$$

Clearly, (3.10), (3.11) and (3.13) can be obtained easily from (3.12). Hence F is of vanishing E-curvature if and only if $\Psi_z = 0$.

4. Projectively flat Finsler metric

Definition 4.1. *A Finsler metric $F = F(x, y)$ on an open subset $U \in \mathbb{R}^n$ is said to be projectively flat if all of its geodesics are straight line in U. A Finsler metric F on a manifold M is said to be a locally projectively flat if at any point, there is a local co-ordinate system (x^i) in which F is projectively flat.*

A Finsler metric F is projectively flat if and only if the spray coefficients can be written in the form $G^i = Py^i$, where P is a scalar function $U \in \mathbb{R}^n \to U \subseteq \mathbb{R}^n$ of x and y.

For projectively flat,

$$G^i = Py^i$$

From equation (2.2) we have, $W = 0$. Which implies $\phi_\rho = 0$. And therefore, F is a Riemannian metric.

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