
AUT Journal of Mathematics and Computing

AUT J. Math. Com., 3(1) (2022) 17-26
DOI: 10.22060/AJMC.2021.19680.1050

Original Article

# Dym equation: group analysis and conservation laws

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**ABSTRACT:** In this paper group-invariant properties of the Dym equation are studied. Lie symmetries are given and some group-invariant solutions are found with the use of similarity variables obtained from these operators. Conservation laws are computed via three methods. Direct method for construction of conservation laws is introduced by the concept of multipliers and Euler-Lagrange operator. Next, the non-linearly self-adjointness of the considered PDE is stated. Then, the modified Noether's theorem is used for finding conservation laws. Finally, the third method is established via the Hereman-Pole method by using the evolutionary form of the equation.

**Review History:**

Received:27 February 2021
Accepted:12 May 2021
Available Online:01 February 2022

**Keywords:**

Dym equation
Non-linear self-adjointness
Lie point symmetries
Conservation laws

**AMS Subject Classification (2010):**

70S10; 76M60

## 1. Introduction

In the theory of solitons, the Dym equation is the third-order PDE

$$u_t - u^3 u_{xxx} = 0, \tag{1}$$

first appeared in Kruskal and is attributed to an unpublished paper by Harry Dym [13, 24, 26]. This equation represents a system in which dispersion and non-linearity are coupled together. Eq. (1) is a completely integrable non-linear evolution equation that may be solved by means of the inverse scattering transform. It is interesting because it obeys an infinite number of conservation laws; it does not possess the Painlevé property. The Dym equation has strong links to the KdV equation. C.S. Gardner, J.M. Greene, Kruskal and R.M. Miura applied [Dym equation] to the solution of corresponding problem in KdV equation. The Lax pair of the Harry–Dym equation is associated with the Sturm–Liouville operator. The Liouville transformation transforms this operator isospectrally into the Schrödinger operator [5]. Thus by the inverse Liouville transformation solutions of the Korteweg–de Vries equation are transformed into solutions of the Dym equation. An explicit solution of the Dym equation, valid in a finite interval, is found by an auto-Bäcklund transform [5].

In physics, a conservation law states that a particular measurable property of an isolated physical system does not change as the system evolves over time. Exact conservation laws include conservation of energy, conservation of linear momentum, conservation of angular momentum, and conservation of electric charge. Also in mathematics conservation law of a given system of DEs is a divergence expression that vanishes on all solutions of the DE system.

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In the study of systems of DEs, the concept of a conservation law plays a very important role in the analysis of essential properties of the solutions, particularly, investigation of existence, uniqueness and stability of solutions [31]. There are several methods for finding conservation laws of a given system of DEs. In this work Ibragimov's method (modified version of Noether's theorem), direct method and Hereman-Pole method are considered.

Lie symmetry operators of the equation are computed for finding the group-invariant solutions. Also these operators will be used in Ibragimov's method in order to calculate conservation laws.

The paper is outlined in four sections. Below of the introduction, in section 2, Lie symmetry operators are found by use of the standard invariant condition. Section 3 is devoted for finding group-invariant solutions obtained from symmetries. Finally in the fourth section the above triple methods are applied in order to obtain conservation laws of the Eq. (1).

## 2. Lie symmetries of Dym equation

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations [10] are solved with the aid of this method. There are still many authors using this method to find the exact solutions [22, 32] of non-linear DEs. It is also a powerful tool for finding exact solutions of non-linear problems [28, 27]. One of the most important application of symmetry's method is to reduce a systems of DEs, i.e., finding equivalent systems of DEs of simpler form, that is called reduction. This method provides a systematic and computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system. Many examples of applications to physical problems have been demonstrated in a huge number of papers and a lot of excellent books. The general procedure to obtain Lie symmetries of DEs, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject [2, 3, 10, 28, 27] and in numerous papers in the literature (e.g. [7, 9, 8, 12, 23, 35, 34]).

Fractional differential equations (FDEs) are a fast developing area of mathematical investigations, both the theory and their applications. During the last four decades, several analytical and numerical methods were presented for solving FDEs [1]. There are several methods an papers on the modeling and solutions of FDEs [4, 6, 14, 15, 16, 17, 18, 19, 20, 33].

Nowadays the group theory of DEs is extended to DEs of fractional order. This subject is the rapidly growing field of research. In recent years, fractional order DEs have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasicity, biology, probability, mathematical physics and engineering [30, 25, 29].

First of all, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned} x &\rightarrow x+\varepsilon\xi(x,t,u),\\ t &\rightarrow t+\varepsilon\tau(x,t,u),\\ u &\rightarrow u+\varepsilon\eta(x,t,u), \end{aligned}$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the above group of transformations can be written as:

$$X=\xi^1(t,x,u)\frac{\partial}{\partial t}+\xi^2(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u}. \tag{2}$$

The symmetry group of Eq. (1) will be generated by the vector field of the form (2). Thus, this equation admits $X$ as a symmetry operator if the condition,

$$X^{(3)}(1)\Big|_{(1)}=0, \tag{3}$$

holds for the third prolongation,

$$X^{(3)}=\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\eta_t\frac{\partial}{\partial u_t}+\eta_x\frac{\partial}{\partial u_x}+\eta_{tt}\frac{\partial}{\partial u_{tt}}+\eta_{tx}\frac{\partial}{\partial u_{tx}}+\eta_{xx}\frac{\partial}{\partial u_{xx}}+\eta_{xxx}\frac{\partial}{\partial u_{xxx}}+\cdots.$$

Applying the third prolonged vector field and solving the determining equation [28, 27], one can demonstrate the Eq. (1) admits the following Lie algebra of symmetries:

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u},\quad X_4=t\frac{\partial}{\partial t}-\frac{u}{3}\frac{\partial}{\partial u},\quad X_5=\frac{x^2}{2}\frac{\partial}{\partial x}+xu\frac{\partial}{\partial u}. \tag{4}$$

## 3. Similarity reductions and exact solutions

In this section symmetry reductions of Eq. (1) will be obtained by means of similarity transformations. Determination of similarity solutions of the Dym (and other) equation is a standard procedure to be found in many texts [21].

- Similarity solution of $X_1$

  For the generator $X_1 = \frac{\partial}{\partial t}$ we have,

$$u = v(q, r), \tag{5}$$

  as a new variable where $q = t$, $r = x$ are the group-invariants. Substituting (5) into (1), one can get,

$$v^3 v''' = 0. \tag{6}$$

  Consequently, the exact solution of Eq. (1) can be written as follows:

$$u(x,t) = d,$$
$$u(x,t) = -\frac{1}{2}ax^2 + bx + c,$$

  where $a, b, c, d$ are arbitrary constants.

- Similarity solution of $X_2$

  The generator $X_2 = \frac{\partial}{\partial x}$ yields the group-invariant $q = x$ and $r = t$. After substitution (5) into (1), we obtain:

$$v' = 0. \tag{7}$$

  Thus, the new exact solution is

$$u(x,t) = at + b$$

  for arbitrary constants.

- Similarity solution of $X_1 + X_2$

  For the sum of two generators $X_1$ and $X_2$, we have $u = v(r, q)$ where $q + r = t$, $q = x$ are the group-invariants. This substitution yields the reduced form

$$v' + v^3 v''' = 0. \tag{8}$$

  So the Eq. (1) was converted to ODE (8). Consequently the solution of this equation satisfies the following identity:

$$\int_a^{u(x,t)} \frac{\varphi}{\sqrt{\varphi(-1 - C_2^2\varphi + C_1\varphi^2 - 2C_2\varphi)}} d\varphi - t + x - C_3 = 0.$$

- Similarity solution of $X_1 + X_3$

  The sum of two generators $X_1$ and $X_3$, gives $u = v(r, q)$ where $q + r = t$, $e^q = x$ are the group-invariants. Substituting $v(r, q)$ into (1), the following reduced equation is obtained:

$$v^3 v' - v^3 v''' - v' = 0. \tag{9}$$

  So the Eq. (25) was reduced to ODE (9). Consequently the solution of this equation satisfies the following identity:

$$\int_a^{u(t+\ln x)} \frac{\varphi}{\sqrt{-\varphi(1 + 2C_1\varphi^2 - 2C_2\varphi - \varphi^3)}} d\varphi - t - \ln x + C_3 = 0.$$

- Traveling wave solutions

  The most useful solution is the traveling wave solution associated with the space and time translation symmetries. Using the transformation,

$$u(x,t)=v(\xi), \quad \xi=x-ct, \tag{10}$$

  and inserting the expression (10) into (1) yields,

$$-Cv'-v^3v'''=0. \tag{11}$$

  Thus, the corresponding traveling wave solution is

$$\int_a^{u(x-ct)} \frac{C\varphi}{\sqrt{C\varphi(C_1\varphi^2-C^2-2C^2C_2\varphi-C^2{C_2}^2\varphi^2)}}d\varphi-x-C_3=0.$$

## 4. Conservation Laws

In the study of PDEs, conservation laws have many significant uses. They describe physical conserved quantities such as energy, momentum and angular momentum. They are used in the analysis of stability and global behavior of solutions. In addition, they play an essential role in the development of numerical methods and provide an essential starting point for finding non-locally related systems and potential variables [23]. In this section we will try to obtain conservation laws of Dym equation in three ways.

### *4.1. The direct method for construction of conservation laws*

In general, non-trivial local conservation laws arise from linear combinations of the equations of the PDEs system with multipliers that yield non-trivial divergence expressions. In asking such expressions, the dependent variables and each of their derivatives that arise in PDE system, or appear in the multipliers, are replaced by arbitrary functions. By their construction, such divergence expressions vanish on all solutions of the PDE system. In particular, a set of multipliers

$$\{\xi_\sigma[U]\}_{\sigma=1}^N=\{\xi(x,U,\partial U,\cdots,\partial^l U)\}_{\sigma=1}^N,$$

yields a divergence expressions for PDEs system $F\{x;u\}$ if the identity

$$\xi_\sigma[U]F^\sigma[U]\equiv D_i\Phi^i[U],$$

holds for arbitrary functions $U(x)$.

A set of non-singular local multipliers $\{\xi_\sigma(x,U,\partial U,\cdots,\partial^l U)\}_{\sigma=1}^N$ yields a local conservation law for the PDEs system $F\{x,u\}$ if and only if the set of identities,

$$E_{U^j}(\xi(x,U,\partial U,\cdots,\partial^l U)F^\sigma(x,U,\partial U,\cdots,\partial^k U))\equiv 0, \tag{12}$$

holds for arbitrary functions $U(x)$, [3]. We apply this method to obtain the local conservation laws of Eq. (1). First all local conservation law multipliers of the zeroth order

$$\xi=\xi(x,t,U), \tag{13}$$

are investigated. Using the Euler operator,

$$\begin{aligned}E_U=\frac{\partial}{\partial U}-D_t\frac{\partial}{\partial U_t}-D_x\frac{\partial}{\partial U_x}+D_x^2\frac{\partial}{\partial U_{xx}}+D_t^2\frac{\partial}{\partial U_{tt}}+D_{xt}\frac{\partial}{\partial U_{xt}}\\ -D_{xtt}\frac{\partial}{\partial U_{xtt}}-D_{xxt}\frac{\partial}{\partial U_{xxt}}-D_{ttt}\frac{\partial}{\partial U_{ttt}}-D_{xxx}\frac{\partial}{\partial U_{xxx}},\end{aligned}$$

yields the following four sets of local multipliers,

$$\xi_1(x,t,u)=\frac{1}{u^3}, \quad \xi_2(x,t,u)=\frac{1}{u^2}, \quad \xi_3(x,t,u)=\frac{x}{u^3}, \quad \xi_4(x,t,u)=\frac{x^2}{2u^3}. \tag{14}$$

Similarly each $\xi$ determines a non-trivial zeroth order local conservation law

$$D_t\Psi(x,t,U)+D_x\Phi(x,t,U)=0,$$

with the characteristic from

$$\xi(x,t,U)F[U]=D_t\Psi(x,t,U)+D_x\Phi(x,t,U). \tag{15}$$

Inserting (14) in (15) for $\xi$, densities and fluxes will be computed. The results are coming in Table 1.

Table 1: Zeroth order local conservation laws

| Density | Flux |
|---|---|
| 1 | 0 |
| $x$ | 0 |
| 0 | 1 |
| $-t$ | $x$ |

Now we seek all local conservation law multipliers of the first order,

$$\xi = \xi(x, t, U, U_x, U_t), \tag{16}$$

of the Eq. (1), with using the corresponding Euler operators. The determining equations (12) for the multipliers (16) become:

$$E_U\left[\xi(x, t, U, U_x, U_t)((U_t - U^3 U_{xxx})\right] \equiv 0. \tag{17}$$

Equations (17) split with respect to each of dependent variables derivatives that arise in PDEs system (except dependent variables and first order derivatives of them). Then the solutions of (17) are the same as given by given by (14). Each $\xi(x, t, U, U_x, U_t)$ determines a non-trivial first order local conservation law

$$D_t\Psi(x, t, U, U_x, U_t) + D_x\Phi(x, t, U, U_x, U_t) = 0,$$

with the characteristic from,

$$\xi(x, t, U, U_x, U_t)F[U] = D_t\Psi(x, t, U, U_x, U_t) + D_x\Phi(x, t, U, U_x, U_t). \tag{18}$$

After replacement (16) in (18) and doing some tedious calculations Table 2 is obtained.

Table 2: First order local conservation laws

| Density | Fluxes |
|---|---|
| $-xu_x - u + 1$ | $xu_t$ |
| $-tu_x - 2tx$ | $tu_t + u + x^2$ |
| $t^2 + x^2$ | $-2xt$ |
| $-u_x - t$ | $u_t + x$ |
| $-uu_x$ | $uu_t + t$ |

Finally the multiplier

$$\xi = \xi(x, t, U, U_x, U_t, U_{xx}, U_{tt}, U_{xt}),$$

is applied for finding the second order conservation law. We can find $\xi$ with the same expression such as (16), and (13). This yields the second order fluxes and densities those are coming in Table 3.

Table 3: Second order local conservation laws

| Density | Fluxes |
|---|---|
| $-u_{tx} - u_{xx}$ | $u_{tt} + u_{xt}$ |
| $-u_t u_{tx}$ | $u_t u_{tt}$ |
| $+uu_{xx} + u_x^2$ | $-uu_{tx} - u_t u_x$ |
| $+uu_{tx} + u_x u_t$ | $-uu_{tt} - u_t^2$ |
| $+xu_x + u + tu_x$ | $-xu_t - tu_t - u$ |
| $-u_x u_{xx} + x$ | $u_x u_{tx} + t$ |
| $tu_{tx} - u_x$ | $-tu_{tt}$ |
| $xu_{tx} + u_t$ | $-xu_{tt}$ |
| $u_t u_{xx} + u_x u_{xt}$ | $-u_x u_{tt} - u_t u_{xt}$ |
| $xu_{xx} + u_x + tu_{xx}$ | $-xu_{tx} - tu_{tx} - u_x$ |
| $-uu_x - 2xt - \frac{t^2}{2}$ | $uu_t + x^2 + xt + t^2 + t^3$ |

### *4.2. Conservation laws provided by Hereman-Pole method*

To compute fluxes and densities for all of coefficients $\xi$, we use two-dimensional homotopy operator $(\mathcal{H}^x_{u(x,t)}(f), \mathcal{H}^t_{u(x,t)}(f))$.

**Definition 4.1.** *Let $f = f(x; u^M(x))$ be a differential function of order $M$. $f$ is called exact if there exists a differential function $F(x; u^{M-1}(x))$ such that $f = D_x F$ and $f = f((x,t); u^M(x,t))$ is exact if there exists a differential vector function $F = f((x,t); u^M(x,t))$ such that $f = DivF$.*

**Theorem 4.1.** *A differential function $f = f(X; U^M(X))$ is exact if and only if $L_{U(X)} f \equiv 0$. Here, 0 is the vector $(0, 0, \cdots, 0)$ which has $N$ components matching the number of components of $U$.*

**Definition 4.2.** *Let $f(X; u^M(X))$ be an exact differential function involving two independent variables $X = (x,t)$. The second homotopy operator is a vector operator with two components, $(\mathcal{H}^x_{u(x,t)}(f), \mathcal{H}^t_{u(x,t)}(f))$ where,*

$$\mathcal{H}^{(x)}_{u(x,t)}(f) = \int_0^1 \sum_{j=1}^{q} \mathcal{I}^{(x)}_{u^j(x,t)}(f)[\lambda u] \frac{d\lambda}{\lambda}, \tag{19}$$

*and,*

$$\mathcal{H}^{(t)}_{u(x,t)}(f) = \int_0^1 \sum_{j=1}^{q} \mathcal{I}^{(t)}_{u^j(x,t)}(f)[\lambda u] \frac{d\lambda}{\lambda}. \tag{20}$$

The $x$-integrand, $\mathcal{I}^{(x)}_{u^j(x,t)} f$, is given by,

$$\mathcal{I}^{(x)}_{u^j(x,t)}(f) = \sum_{k_1=1}^{M_1^j} \sum_{k_2=0}^{M_2^j} \left[ \sum_{i_1=0}^{k_1-1} \sum_{i_2=0}^{k2} B^x u_{x^{i_1} t^{i_2}} (-D_x)^{k_1-i_1-1} (-D_t)^{k_2-i_2} \right] \frac{\partial f}{\partial u_{x^{k_1} t^{k_2}}}, \tag{21}$$

with combinatorial coefficient $B^x = B(i_1, i_2, k_1, k_2)$ expressed as:

$$B^x = \frac{\binom{i_1+i_2}{i_1} \binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+k_2}{k_1}}. \tag{22}$$

The $t$-integrand, $\mathcal{I}^{(t)}_{u^j(x,t)} f$ is defined as:

$$\mathcal{I}^{(t)}_{u^j(x,t)}(f) = \sum_{k_1=0}^{M_1^j} \sum_{k_2=1}^{M_2^j} \left[ \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k2-1} B^t u_{x^{i_1} t^{i_2}} (-D_x)^{k_1-i_1} (-D_t)^{k_2-i_2-1} \right] \frac{\partial f}{\partial u_{x^{k_1} t^{k_2}}}, \tag{23}$$

with combinatorial coefficient $B^t = B(i_1, i_2, k_1, k_2)$ expressed as:

$$B^t = \frac{\binom{i_1+i_2}{i_2} \binom{k_1+k_2-i_1-i_2-1}{k_2-i_2-1}}{\binom{k_1+k_2}{k_1}}. \tag{24}$$

Also $M_1^j$ is the order of $f$ in dependent variable $u_j$ with respect to $x$, and $M_2^j$ is the order of $f$ in dependent variable $u_j$ with respect to $t$. The notation $f[\lambda u]$ means that in $f$ one replaces $u$ by $\lambda u$ and $u_x$ by $\lambda u_x$, and so on for all derivatives of $u$, that $\lambda$ is an auxiliary parameter. All the results for homotopy operators and related conservation laws are coming in Table 4.

### *4.3. Conservation laws provided by Ibragimov's method*

First, the meaning of non-linear self-adjointness should be stated. Let us

$$F = u_t - u^3 u_{xxx}. \tag{25}$$

The formal Lagrangian for the equation (25) is given by

$$\mathcal{L} = v\big(u_t - u^3 u_{xxx}\big), \tag{26}$$

Table 4: Second order local conservation laws

| $\xi_i$ | $I^t$ | $I^x$ | $H^t f$ | $H^x f$ |
|---|---|---|---|---|
| $\frac{1}{u^3}$ | $\frac{1}{u^2}$ | $-u_{xx}$ | $-\frac{1}{2u^2}$ | $-u_{xx}$ |
| $\frac{1}{u^2}$ | $\frac{1}{u}$ | $-2uu_{xx}+u_x^2$ | $-\frac{1}{u}$ | $\frac{1}{2}u_x^2-uu_{xx}$ |
| $\frac{x}{u^3}$ | $\frac{x}{u^2}$ | $u_x-xu_{xx}$ | $-\frac{x}{2u^2}$ | $u_x-xu_{xx}$ |
| $\frac{x^2}{2u^3}$ | $\frac{x^2}{2u^2}$ | $-u+xu_x-\frac{x^2}{2}u_{xx}$ | $-\frac{x^2}{4u^2}$ | $-u+xu_x-\frac{x^2}{2}u_{xx}$ |
| | | **for the second order** | | |
| $\frac{x}{u^3}$ | $\frac{x}{u^2}$ | $u_x-xu_{xx}$ | $-\frac{x}{2u^2}$ | $u_x-xu_{xx}$ |
| $-\frac{1}{3u^3}$ | $-\frac{1}{3u^2}$ | $\frac{u_{xx}}{3}$ | $\frac{1}{6u^2}$ | $\frac{u_{xx}}{3}$ |
| $-\frac{1}{2u^2}$ | $-\frac{1}{2u}$ | $-\frac{1}{2}u_x^2+u_{xx}u$ | $\frac{1}{2u}$ | $-\frac{1}{2}u_x^2$ |
| $\frac{x^2}{2u^3}$ | $\frac{x^2}{2u^2}$ | $-u+xu_x-\frac{x^2}{2}u_{xx}$ | $-\frac{x^2}{4u^2}$ | $-u+xu_x-\frac{x^2}{2}u_{xx}$ |

where $v=\varphi(u)$ is new dependent variables. The adjoint equation to system (25) is determined by,

$$F^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \tag{27}$$

where $\delta/\delta u$ is the variational derivative:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}} - D_i D_j D_k \frac{\partial}{\partial u_{ijk}} + \cdots .$$

We will identify the first and the second independent variables with the time-like variable $t$ and space-like variable $x$, respectively then the total derivatives $D_i$ have the form,

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tj} \frac{\partial}{\partial u_j} + u_{tjk} \frac{\partial}{\partial u_{jk}} + \cdots , \\ D_x &= \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial u} + u_{xj} \frac{\partial}{\partial u_j} + u_{xjk} \frac{\partial}{\partial u_{jk}} + \cdots . \end{aligned} \tag{28}$$

Now the expanded form of the adjoint system (27) is

$$\left.\frac{\delta \mathcal{L}}{\delta u}\right|_{u=v} = \frac{1}{u^4}\left(u_t - u^3 u_{xxx}\right), \tag{29}$$

thus, the new variable is

$$v = \frac{1}{u^4}. \tag{30}$$

It means that the system (25) is non-linearly self-adjoint, specifically it is quasi self-adjoint. Now, we obtain conservation laws for the Dym equation via Ibragimov's theorem [11].

**Theorem 4.2.** *Every Lie point, Lie-backlund and non-local symmetry of Eq. (1) provides a conservation law for Eq. (1) and the adjoint equation. Then the elements of conservation vector $(C^1, C^2)$ are given by,*

$$\begin{aligned} C^i &= \xi^i \mathcal{L} + W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_i D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \cdots \right] \\ &\quad + D_j(W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\ &\quad + D_{jk}(W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - D_s \left( \frac{\partial \mathcal{L}}{\partial u_{ijks}^\alpha} \right) + \cdots \right] + \cdots , \end{aligned} \tag{31}$$

*where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.*

The corresponding conserved components for Dym equation are

$$\begin{aligned} C^1 &= \xi^1 \mathcal{L} + Wv, \\ C^2 &= \xi^2 \mathcal{L} + W(D_x D_x(-u^3 v)) + D_x(W)(-D_x(-u^3 v)) + D_x D_x(W)(-u^3 v), \end{aligned}$$

with $W = \eta - \xi^1 u_t - \xi^2 u_x$. After replacement (30) in (31) Table 5 is obtained.

Table 5: Local Conservation Laws via Ibragimov's Method

| vector field | $C^1$ | $C^2$ |
|---|---|---|
| $X_1$ | $\frac{-u_t}{u^4}$ | $\frac{2u_t u_x^2}{u^3} + \frac{u_x u_{xt}}{u^2} + \frac{u_{xxt}}{u} - \frac{u_t u_{xx}}{u^2}$ |
| $X_2$ | $\frac{-u_x}{u^4}$ | $\frac{2u_x^3}{u^3} + \frac{u_{xxx}}{u}$ |
| $X_3$ | $\frac{1}{u^3} - \frac{xu_x}{u^4}$ | $\frac{2u_x^2}{u^2} + \frac{2u_{xx}}{u} - \frac{2xu_x^3}{u^3} + \frac{xu_t}{u^4}$ |
| $X_4$ | $\frac{-1}{3u^3} - \frac{tu_t}{u^4}$ | $\frac{2u_x^2}{3u^2} - \frac{u_{xx}}{3u} + \frac{2tu_x^2 u_t}{u^3} - \frac{tu_t u_{xx}}{u^2} + \frac{u_x^2}{3u^2} + \frac{tu_{xt} u_x}{u^2} + \frac{u_x u}{3} + \frac{tu_{xxt}}{u}$ |
| $X_5$ | $\frac{x}{u^3} - \frac{x^2 u_x}{2u^4}$ | $\frac{2xu_{xx}}{u} - \frac{2u_x}{u} - \frac{2xu_x^2}{u^2} + \frac{x^2 u_x^3}{u^3} + \frac{x^2 u_{xxx}}{2u}$ |

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Please cite this article using:

S. Reza Hejazi, Azadeh Naderifard, Dym equation: group analysis and conservation laws,AUT J. Math. Com., 3(1) (2022) 17-26
DOI: 10.22060/AJMC.2021.19680.1050