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Original Article

Extracting some supra topologies from the topology of a topological space using stacks

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ABSTRACT: A collection μ of subsets of a nonempty set X is a supra topology on X whenever $\emptyset$ and X belong to μ, and also μ is closed under arbitrary unions. Also, a nonempty collection $\mathcal{S}$ of nonempty subsets of a nonempty set X is called a stack on X whenever it is closed under operation superset. In this paper, we are going to introduce an approach to extract some supra topologies from the topology of a topological space. For this purpose, we consider a topological space (X, τ) with a closed set P of its subsets. Using a stack $\mathcal{S}$ on the space (X, τ) and the closure operator cl associated with τ, we define a supra closure operator λ_P on X to create the desired supra topology. We then characterize the form of this resulting supra topology and also determine its relationship to the initial topology of the space.

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1. Introduction and preliminary

Some structures of general topology-as an abstract branch of mathematics- such as topology, generalized topology, supra topology and proximity structure can be used in many diffrent sciences, including IT and computer science(e.g. digital topology, digital plane, digital surface and digital manifold). If the arrangement of information of a number of data can be defined as a topological structure, then applying a new condition(or conditions) on the data information can lead to change the previous arrangement which can be considered as change in that topological structure. Affected by this issue, in this paper we intend to change the topology of a topological space to supra topology. Removing the finite intersection condition of topological spaces, a structure called supra topology was introduced in 1983 by Mashhour et al. [7].

Definition 1.1. *[7] Let X be a nonempty set and μ be a nonempty collection of subsets of X. μ is called a supra topology (briefly ST) on X whenever $\emptyset, X \in \mu$ and $\{G_\alpha\}_{\alpha\in\Lambda} \subseteq \mu$ implies $\cup_{\alpha\in\Lambda} G_\alpha \in \mu$. The pair (X, μ) is called a supra topological space (briefly STS). The members of μ are then called supra-open sets, and also a subset of a supra topological space (X, μ) is called a supra-closed set if its complement is a supra-open set.*

Although some topological features are not true of supra topologies (e.g. the intersection of two supra-open sets need not be supra-open, see [7]), but many various notions like interior, closure, compactness, connectedness and

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etc. can be define in a supra topological space in analogy with topological spaces. Al-Shami [1] has studied the classical topological notions such as limit points of a set, compactness, and separation axioms on the supra topological spaces. Also, supra α-open [4], supra preopen [8], supra b-open [9], supra β-open [6], supra R-open [5], and supra semi-open sets [2] have been introduced and their main properties have been discussed. So it seems that super spaces are rich in structure. Given the importance and richness of supra spaces, in this paper we intend to provide a way to extract some kinds of supra spaces from topological spaces. First let's start with the following definition.

Definition 1.2. *[7] Denote $\mathcal{P}(X)$ as the power set of X. Then λ as an operator on $\mathcal{P}(X)$ is called a supra closure operator if it satisfies the following axioms;*

1. *$\lambda(\emptyset) = \emptyset$ (property of nullity),*
2. *$A \subseteq \lambda(A)$ for any $A \subseteq X$ (property of extensivity),*
3. *for any $A, B \subseteq X$, $A \subseteq B$ implies that $\lambda(A) \subseteq \lambda(B)$ (property of monotonicity),*
4. *$\lambda(\lambda(A)) = \lambda(A)$ for any $A \subseteq X$ (property of idempotency).*

From [7] it is well-known that supra topological spaces are characterized by supra closure operator. In fact, associated with any supra topology μ on a set X is a supra closure operator on the set X, denoted λ_μ or λ_X(in short, λ), which gives for any subset $A \subseteq X$, the smallest supra closed set $\lambda(A)$ containing A. Also, on the other hand, corresponding to any supra closure operator λ on a set X, there exists a unique supra topology, say, μ on the set X in the form of $\mu = \{X - A : \lambda(A) = A\}$.

In this paper, we are going to introduce a method based on which we can turn the topology of a desired topological space (X, τ) into a supra topology, say μ, so in fact we change a topological space (X, τ) to a supra topological space (X, μ).

The method of work is based on that by choosing a stack $\mathcal{S}$ on an arbitrary space (X, τ) and the use of the Kuratowski closure operator cl corresponding to the topology of this space we define a supra closure operator λ_P, where P is a closed subset of (X, τ), to extract the desired ST.

Here, let us to provide a brief description of the method.

Let (X, τ) be an arbitrary topological space, P and $\mathcal{S}$ be respectively, an arbitrary closed subset of (X, τ) and a stack on (X, τ). Then we define an operator λ_P on X based on the stack $\mathcal{S}$ and the closure operator cl associated with τ, as follows;

$$\lambda_P^{\mathcal{S}}(A) = \begin{cases} clA & clA \notin \mathcal{S} \\ clA \cup P & clA \in \mathcal{S} \end{cases} \tag{1}$$

where $A \in \mathcal{P}(X)$. For simplicity, we use λ_P instead of $\lambda_P^{\mathcal{S}}$, provided there is no ambiguity.

ln the next section, it will be shown that the operator λ_P satisfies supra closure operator axioms, that is, the operator λ_P has the quadratic properties of nullity, extensivity, monotonicity and idempotency, so according to [7] it can constitute a supra topology, say μ_P on X. We then completely determine the form of this supra topology and show that $\mu_P \subseteq \tau$. Also, using several different stacks, we provide some examples of μ_P.

Throughout the paper, let us use the symbol of X^Y as the collection of all operators from set X to set Y, and (X, τ) as an arbitrary topological space. Also, $\mathcal{S}(X)$ stands on the collection of all stacks on X and we will represent the complement of a set A to the whole set X with the symbols $X \setminus A$ or $X - A$.

2. Main results

As proposed in the introduction, this section intends to provide a method to extract some supra topologies from the topology of a topological space. To that end, we first give the general construction of a supra closure operator from the Kuratowski closure operator associated with an arbitrary topological space.

Lemma 2.1. *Let (X, τ) be a topological space. Also, let P and $\mathcal{S}$ be respectively, a closed subset of (X, τ) and a stack on (X, τ). Then the operator $\lambda_P \in \mathcal{P}(X)^{\mathcal{P}(X)}$ defined by*

$$\lambda_P(A) = \begin{cases} clA & clA \notin \mathcal{S} \\ clA \cup P & clA \in \mathcal{S} \end{cases}$$

for any $A \subseteq X$, has the following properties;

For two subsets A and B of X;

(i): if $cl(A \cup B) \notin \mathcal{S}$, then $\lambda_P(A \cup B) = \lambda_P(A) \cup \lambda_P(B)$.

(ii): In case of $cl(A\cup B) \in \mathcal{S}$, if at least one of clA or clB belongs to $\mathcal{S}$, then we get $\lambda_P(A\cup B) = clA\cup clB\cup P = \lambda_P(A) \cup \lambda_P(B)$, but if both clA and clB do not belong to $\mathcal{S}$, $\lambda_P(A \cup B)$ cannot be equal to $\lambda_P(A) \cup \lambda_P(B)$, in general.

Proof. Let A and B are two subsets of X;

(i): As $cl(A \cup B) \notin \mathcal{S}$ we have $\lambda_P(A \cup B) = cl(A \cup B) = clA \cup clB$.

Again from $cl(A \cup B) \notin \mathcal{S}$ we have $clA, clB \notin \mathcal{S}$. So, $clA \cup clB = \lambda_P(A) \cup \lambda_P(B)$. Therefore in this case we have;

$$\lambda_P(A \cup B) = clA \cup clB = \lambda_P(A) \cup \lambda_P(B).$$

(ii): Let $cl(A \cup B) \in \mathcal{S}$, then $\lambda_P(A \cup B) = clA \cup clB \cup P$.

Now, if $clA, clB \in \mathcal{S}$, then we have, $\lambda_P(A) = clA \cup P$ and $\lambda_P(B) = clB \cup P$. So, we get

$$\lambda_P(A \cup B) = cl(A \cup B) \cup P = (clA \cup P) \cup (clB \cup P) = \lambda_P(A) \cup \lambda_P(B).$$

Also, in this case when only one of clA or clB belongs to $\mathcal{S}$ and without loss the generality if $clA \in \mathcal{S}$ but $clB \notin \mathcal{S}$, then we have $\lambda_P(A \cup B) = clA \cup clB \cup P$ and also, $\lambda_P(A) = clA \cup P$ and $\lambda_P(B) = clB$. So, we get

$$\lambda_P(A \cup B) = cl(A \cup B) \cup P = (clA \cup P) \cup clB = \lambda_P(A) \cup \lambda_P(B).$$

But, in this case if $clA, clB \notin \mathcal{S}$, then by choosing $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, X\}$, $\mathcal{S} = \{\{a, b, c\}, X\}$ and $P = \{b, c, d\}$, for $A = \{a, c\}$ and $B = \{b, c\}$, we will have $clA = \{a, c\} \notin \mathcal{S}$ and $clB = \{b, c\} \notin \mathcal{S}$. So, $\lambda_P(A) = clA = \{a, c\}$ and $\lambda_P(B) = clB = \{b, c\}$ and thus $\lambda_P(A) \cup \lambda_P(B) = \{a, b, c\}$. But as $cl(A \cup B) = \{a, b, c\} \in \mathcal{S}$, we have $\lambda_P(A \cup B) = cl(A \cup B) \cup P$. So, for this case

$$\lambda_P(A \cup B) = cl(A \cup B) \cup \{b, c, d\} = \{a, b, c\} \cup \{b, c, d\} = \{a, b, c, d\} \neq \{a, b, c\} = \lambda_P(A) \cup \lambda_P(B).$$

□

The following theorem shows that the operator λ_P is a supra closure operator.

Theorem 2.2. *Let (X, τ) be a topological space and P be a closed subset of it. Also, let $\mathcal{S}$ be a stack on (X, τ). Then the operator $\lambda_P \in \mathcal{P}(X)^{\mathcal{P}(X)}$ for any $A \subseteq X$, defined by*

$$\lambda_P(A) = \begin{cases} clA & clA \notin \mathcal{S} \\ clA \cup P & clA \in \mathcal{S} \end{cases}$$

is a supra closure operator, inducing a supra topology μ_P on X.

Proof. It is clear that $\lambda_P(\emptyset) = \emptyset$.
clearly, for any $A \subseteq X$ we have $A \subseteq \lambda_P(A)$, that is, λ_P is an extensive operator. Also, obviously $\lambda_P(A) \subseteq \lambda_P(B)$ whenever $A \subseteq B$, that is, λ_P has the property of monotonicity.

We next show that $\lambda_P(\lambda_P(A)) = \lambda_P(A)$, for any $A \subseteq X$.

Here, let A be a subset of X and consider two cases;

(i): $clA \notin \mathcal{S}$

(ii): $clA \in \mathcal{S}$

In case (i); we have

$$\lambda_P(\lambda_P(A)) = \lambda_P(clA)\,(since\;\; cl(clA) = clA \notin \mathcal{S}) = clA = \lambda_P(A),$$

while in case (ii); we have

$\lambda_P(\lambda_P(A)) = \lambda_P((clA) \cup P)$(due to part (ii) of Lemma 2.1)$= \lambda_P(clA) \cup \lambda_P(P) = ((clA) \cup P) \cup cl(P)$(due to closedness of P in (X,τ))$= (clA) \cup P = \lambda_P(A)$.

It follows that λ_P is a supra closure operator on X, so from [7] it gives rise to a supra topology(say) μ_P on X. □

Remark 2.3. *In the above theorem, if the set P is selected equal to the empty set then, $\lambda_P = \lambda_\emptyset = cl$, so, $\mu_P = \mu_\emptyset = \tau$. Therefore, from now on, we will consider the closed set P as a nonepmty set.*

Similar to duality of topological closure operator cl_τ and topological interior operator int_τ in any topological space (X,τ), we can define the supra interior operator ι_μ as dual of the supra closure operator λ_μ in any supra topological space (X,μ), in the sense of $\iota_\mu(A) = X - \lambda_\mu(X - A)$ and $\lambda_\mu(A) = X - \iota_\mu(X - A)$ for any $A \subseteq X$.

In the following we determine the form of operator ι_P as dual of λ_P.

Theorem 2.4. *Let (X,τ) be a topological space and λ_P be the operator constructed in Theorem 2.2. Then for any subset A of X the supra interior operator ι_P as dual of the supra closure operator λ_P has the form of*

$$\iota_P(A) = \begin{cases} intA & if\ P\cap A \neq \emptyset,\ and\ \ cl(X\setminus A) \notin \mathcal{S} \\ (intA)\setminus P & if\ P\cap A \neq \emptyset,\ and\ \ cl(X\setminus A) \in \mathcal{S} \\ intA & if\ P\cap A = \emptyset \end{cases}$$

Proof. Let A be a subset of X and consider two cases (i): $P \cap A \neq \emptyset$ and (ii): $P \cap A = \emptyset$.

In case (i), considering $cl(X \setminus A) \notin \mathcal{S}$ we have $\lambda_P(X \setminus A) = cl(X \setminus A)$, so

$$\iota_P(A) = X \setminus \lambda_P(X \setminus A) = X \setminus cl(X \setminus A) = intA,$$

while considering $cl(X \setminus A) \in \mathcal{S}$ leads to $\lambda_P(X \setminus A) = cl(X \setminus A) \cup P$, and so

$$\begin{aligned} \iota_P A &= X \setminus \lambda_P(X \setminus A) = X \setminus [cl(X \setminus A) \cup P] \\ &= (X \setminus cl(X \setminus A)) \cap (X \setminus P) = (intA) \setminus P. \end{aligned}$$

In case (ii) we have; $P \subseteq X \setminus A$. Here, considering $cl(X \setminus A) \notin \mathcal{S}$ leads to $\lambda_P(X \setminus A) = cl(X \setminus A)$, so

$$\iota_P(A) = X \setminus \lambda_P(X \setminus A) = X \setminus cl(X \setminus A) = intA,$$

Whereas considering $cl(X \setminus A) \in \mathcal{S}$ gives $\lambda_P(X \setminus A) = cl(X \setminus A) \cup P(\text{as } P \subseteq X \setminus A) = cl(X \setminus A)$ and so

$$\iota_P(A) = X \setminus \lambda_P(X \setminus A) = X \setminus cl(X \setminus A) = intA.$$

Therefore, according to the above we have the following formula

$$\iota_P(A) = \begin{cases} intA & if\ P\cap A \neq \emptyset,\ cl(X\setminus A) \notin \mathcal{S} \\ (intA)\setminus P & if\ P\cap A \neq \emptyset,\ cl(X\setminus A) \in \mathcal{S} \\ intA & if\ P\cap A = \emptyset \end{cases}$$

and we are done. □

Determining the set $\{A \subseteq X : \iota_P(A) = A\}$ as the fixed points of the supra interior operator ι_P, supra-open sets are identified and therefore the supra topology μ_P is determined.

Corollary 2.5. *From results of Therorem 2.4, we have*

$$\mu_P = \mu_P^{\mathcal{S}} = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{S}\}.$$

Proof. To determine μ_P, we note that $A \in \mu_P$ if and only if $\iota_P(A) = A$. Considering $cl(X \setminus A) \notin \mathcal{S}$ in case (i) in the proof of the Therorem 2.4, leads to $\iota_P(A) = intA$, so $A \in \mu_P$ if and only if $A \in \tau$, while considering $cl(X \setminus A) \in \mathcal{S}$ in case (i), leads to $\iota_P(A) = (intA) \setminus P$, so $A \in \mu_P$ if and only if $A = (intA) \setminus P$ and this is impossile. Therefore, no subset A of X, which intersects P and is valid under the condition $cl(X \setminus A) \in \mathcal{S}$, can belong to μ_P. In case (ii) of the proof of the Therorem 2.4, because $\iota_P(A) = intA$, hence $A \in \mu_P$ if and only if $A \in \tau$.
So according to the above we have

$$\mu_P = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{S}\}.$$

□

In the following remark, the relationship between τ and μ_P has been determined.

Remark 2.6. *In general, $\mu_P \subseteq \tau$, because $\tau = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset\}$ and clearly, $\{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{S}\} \subseteq \{A \in \tau : P \cap A \neq \emptyset\}$.*

In the following example, considering nice stacks $\mathcal{I} = \{A \subseteq X : intA \neq \emptyset\}$ and $\mathcal{D} = \{A \subseteq X : clA = X\}$ on a topological space (X,τ), we extract two supra topologies $\mu_P^{\mathcal{I}}$ and $\mu_P^{\mathcal{D}}$ from the topology of (X,τ).

Example 2.1. *Let (X, τ) be a topological space with $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$. Choosing closed subset $P = \{b, c\}$ of (X, τ), we have;*

$$\begin{aligned} \mathcal{I} &= \{A \subseteq X : intA \neq \emptyset\} \\ &= \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{D} &= \{A \subseteq X : clA = X\} \\ &= \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}. \end{aligned}$$

So, according to Corollary 2.5, we get

$$\mu^{\mathcal{I}}_{\{b,c\}} = \{\emptyset, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{a, b, c, d\}\} = \tau$$

and

$$\mu^{\mathcal{D}}_{\{b,c\}} = \{\emptyset, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{a, b, c, d\}\} = \tau.$$

As we see in this example, $\mu^{\mathcal{I}}_{\{b,c\}} = \mu^{\mathcal{D}}_{\{b,c\}}$.

In the next example, $\mu^{\mathcal{I}}_P$ and $\mu^{\mathcal{D}}_P$ are not equal.

Example 2.2. *Let (X, τ) be a topological space that $\tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, X\}$ be a topology on $X = \{a, b, c, d\}$. Then, we have;*

$$\mathcal{I} = \{\{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$$

and

$$\mathcal{D} = \{\{a, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}.$$

So

$$\mu^{\mathcal{I}}_{\{c,d\}} = \{\emptyset, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{a, b, c, d\}\}$$

and

$$\mu^{\mathcal{D}}_{\{c,d\}} = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

Hence, $\mu^{\mathcal{I}}_{\{c,d\}} \subsetneq \mu^{\mathcal{D}}_{\{c,d\}}$.

Definition 2.7. *If $\mathcal{S}$ is any stack on a STS (X, μ), then $dual(\mathcal{S}) = \{A \subseteq X : X \setminus A \notin \mathcal{S}\}$ is called the dual of $\mathcal{S}$.*

Proposition 2.8. *For any stack $\mathcal{S}$ on a STS (X, μ), $dual(\mathcal{S})$ is a stack on X.*

Proof. If we put $A = \emptyset$, then we have $X \setminus A = X \in \mathcal{S}$, so $\emptyset \notin dual(\mathcal{S})$. Now, let $A \subseteq B \subseteq X$ and $A \in dual(\mathcal{S})$, then $X \setminus B \subseteq X \setminus A$ and $X \setminus A \notin \mathcal{S}$, so $X \setminus B \notin \mathcal{S}$ and therefore $B \in dual(\mathcal{S})$. □

In the following lemma, we have specified the aesthetic relationship between two stacks $\mathcal{I}$ and $\mathcal{D}$.

Lemma 2.9. *Let (X, τ) be a topological space. Considering two stacks $\mathcal{I}$ and $\mathcal{D}$ on the space (X, τ), we have;*

1. *$dual(\mathcal{I}) = \mathcal{D}$ and $dual(\mathcal{D}) = \mathcal{I}$,*
2. *$dual(dual(\mathcal{I})) = \mathcal{I}$ and $dual(dual(\mathcal{D})) = \mathcal{D}$,*
3. *for any closed subset P of (X, τ), $\mu^{\mathcal{I}}_P = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ A \in \mathcal{D}\}$, and $\mu^{\mathcal{D}}_P = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ A \in \mathcal{I}\}$.*

Proof. Part (1): According to Definition 2.7;

$$\begin{aligned} dual(\mathcal{I}) &= \{A \subseteq X : X \setminus A \notin \mathcal{I}\} \\ &= \{A \subseteq X : int(X \setminus A) = \emptyset\} = \{A \subseteq X : X \setminus clA = \emptyset\} \\ &= \{A \subseteq X : clA = X\} = \mathcal{D}. \end{aligned}$$

Again from Definition 2.7;

$$\begin{aligned} dual(\mathcal{D}) &= \{A \subseteq X : X \setminus A \notin \mathcal{D}\} \\ &= \{A \subseteq X : cl(X \setminus A) \neq X\} = \{A \subseteq X : X \setminus intA \neq X\} \\ &= \{A \subseteq X : intA \neq \emptyset\} = \mathcal{I}. \end{aligned}$$

Part (2): According to part (1), we have;
$dual(dual(\mathcal{I})) = dual(\mathcal{D}) = \mathcal{I}$ and $dual(dual(\mathcal{D})) = dual(\mathcal{I}) = \mathcal{D}$.
Part (3): From Corollary 2.5, we have;

$$\begin{aligned} \mu_P^{\mathcal{I}} &= \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{I}\} \\ &= \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ A \in dual(\mathcal{I}) = \mathcal{D}\}, \end{aligned}$$

and

$$\begin{aligned} \mu_P^{\mathcal{D}} &= \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{D}\} \\ &= \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ A \in dual(\mathcal{D}) = \mathcal{I}\}. \end{aligned}$$

□

Assuming that $\mathcal{S}$ is a stack on X, we will use the symbol of λ_P^* (respectively, ι_P^*) instead of $\lambda_P^{dual(\mathcal{S})}$ (respectively, $\iota_P^{dual(\mathcal{S})}$) for convenience and simplicity.

Corollary 2.10. *Let (X, τ) be a topological space and P be a closed subset of it. Also, let $\mathcal{S} \in \mathcal{S}(X)$. Then the operator $\lambda_P^* \in \mathcal{P}(X)^{\mathcal{P}(X)}$ associted with the stack $dual(\mathcal{S})$ is a supra closure operator, and for any $A \in \mathcal{P}(X)$ has the form of*

$$\lambda_P^*(A) = \begin{cases} clA & int(X \setminus A) \in \mathcal{S} \\ clA \cup P & int(X \setminus A) \notin \mathcal{S} \end{cases}$$

Proof. From Theorem 2.2 we have;

$$\lambda_P^*(A) = \lambda_P^{dual(\mathcal{S})} = \begin{cases} clA & clA \notin dual(\mathcal{S}) \\ clA \cup P & clA \in dual(\mathcal{S}) \end{cases}$$

But from definition of $dual(\mathcal{S})$ in Definition 2.7;

$$\lambda_P^{dual(\mathcal{S})} = \begin{cases} clA & int(X \setminus A) \in \mathcal{S} \\ clA \cup P & int(X \setminus A) \notin \mathcal{S} \end{cases}$$

□

Corollary 2.11. *Let (X, τ) be a topological space and λ_P^* be the operator constructed in Corollary 2.10. Then for any subset A of X the supra interior operator ι_P^* as dual of the supra closure operator λ_P^* has the form of*

$$\iota_P^*(A) = \iota_P^{dual(\mathcal{S})}(A) = \begin{cases} intA & P \cap A \neq \emptyset \ with \ intA \in \mathcal{S} \\ (intA) \setminus P & P \cap A \neq \emptyset \ with \ intA \notin \mathcal{S} \\ intA & P \cap A = \emptyset \end{cases}$$

Proof. From theorem 2.4, we have;

$$\iota_P^{dual(\mathcal{S})}(A) = \begin{cases} intA & P \cap A \neq \emptyset \ with \ cl(X \setminus A) \notin dual(\mathcal{S}) \\ (intA) \setminus P & P \cap A \neq \emptyset \ with \ cl(X \setminus A) \in dual(\mathcal{S}) \\ intA & P \cap A = \emptyset \end{cases}$$

Now from definition of $dual(\mathcal{S})$ in Definition 2.7, we have; $\forall A \subseteq X\ (A \in dual(\mathcal{S}) \iff X \setminus A \notin \mathcal{S})$, so,

$$\iota_p^*(A) = \begin{cases} intA & P \cap A \neq \emptyset \ with \ X \setminus cl(X \setminus A) = intA \in \mathcal{S} \\ (intA) \setminus \{p\} & P \cap A \neq \emptyset \ with \ X \setminus cl(X \setminus A) = intA \notin \mathcal{S} \\ intA & P \cap A = \emptyset \end{cases}$$

□

Corollary 2.12. *Let (X, τ) be a topological space and P be a closed subset of it. Also, let $\mathcal{S} \in \mathcal{S}(X)$. Then from results of Corollary 2.5, μ_P^* as the supra topology associted with the stack $dual(\mathcal{S})$ has the form of*

$$\mu_P^* = \mu_P^{dual(\mathcal{S})} = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ A \in \mathcal{S}\}.$$

Proof. From Corollary 2.5, we have;

$$\mu_P^{dual(\mathcal{S})} = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin dual(\mathcal{S})\}.$$

Using definition of $dual(\mathcal{S})$ in Definition 2.7, we have; $\forall A \subseteq X$ $(A \in dual(\mathcal{S}) \iff X \setminus A \notin \mathcal{S})$, so,

$$\mu_P^* = \{A \in \tau : P \cap A = \emptyset\} \cup \{A \in \tau : P \cap A \neq \emptyset,\ A \in \mathcal{S}\}.$$

□

Remark 2.13. *Let P be a closed subset of (X, τ), then for any stack $\mathcal{S} \in \mathcal{S}(X)$, we have;*

(1)*: the set $\{A \in \tau : P \cap A = \emptyset\}$ as a part of μ_P is a topology on $X \setminus P$. Because, clearly $\emptyset \in \{A \in \tau : P \cap A = \emptyset\}$ and $X \setminus P$(since we choose P as a closed set of τ) is in $\{A \in \tau : P \cap A = \emptyset\}$.*

Now, let A_1 and A_2 are in $\{A \in \tau : P \cap A = \emptyset\}$, then $A_1 \cap A_2 \in \tau$ that, $P \cap (A_1 \cap A_2) = \emptyset$. So $A_1 \cap A_2 \in \{A \in \tau : P \cap A = \emptyset\}$.

Also if for an arbitrary indexing set Γ, $\{A_\gamma : \gamma \in \Gamma\}$ be a subcollection of $\{A \in \tau : P \cap A = \emptyset\}$, then $\forall \gamma \in \Gamma$ we have $A_\gamma \in \tau$ and $P \cap A_\gamma = \emptyset$. So $\cup_{\gamma\in\Gamma} A_\gamma \in \tau$ and $P \cap (\cup_{\gamma\in\Gamma} A_\gamma) = \emptyset$, hence $\cup_{\gamma\in\Gamma} A_\gamma \in \{A \in \tau : P \cap A_\gamma = \emptyset\}$. Thus, we show that the set $\{A \in \tau : P \cap A_\gamma = \emptyset\}$ forms a topology on $X \setminus P$, but as $X \setminus P$ is an open set in (X, τ), then we infer that the topology $\{A \in \tau : P \cap A_\gamma = \emptyset\}$ is the subspace topology from τ on $X \setminus P$.

(2)*: putting $S = \{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{S}\}$, then $S \cup \{\emptyset\}$ is a supra topology on X. Because, clearly $\emptyset \in S \cup \{\emptyset\}$ and also, as $X \in \tau$, with $P \cap X \neq \emptyset$ and $X \setminus X = \emptyset \notin \mathcal{S}$, we get $X \in S$. On the other hand, if for an arbitrary indexing set Δ we put $\{\emptyset \neq A_\delta : \delta \in \Delta\} \subseteq S$, then clearly $\cup_{\delta\in\Delta} A_\delta \in \tau$ with $P \cap (\cup_{\delta\in\Delta} A_\delta) \neq \emptyset$ and since $X \setminus \cup_{\delta\in\Delta} A_\delta = \cap_{\delta\in\Delta}(X \setminus A_\delta) \subseteq X \setminus A_{\delta_0} \notin \mathcal{S}$ for some $\delta_0 \in \Delta$, so $X \setminus \cup_{\delta\in\Delta} A_\delta \notin \mathcal{S}$. Therefore $\cup_{\delta\in\Delta} A_\delta \in S$.*

(3)*: Although the set $\{A \in \tau : P \cap A \neq \emptyset,\ X \setminus A \notin \mathcal{S}\} \cup \{\emptyset\}$ expressed in part (2) is a supra topology on X, but this supra topology is not derived from any λ_P (for this reason in the paper, we have used the word "some" in the title of "Extracting some supra topologies from the topology of a topological space using stacks"). Because, here we have $\{A \in \tau : P \cap A = \emptyset\} = \{\emptyset\}$ which means the set P is dense in (X, τ), while the necessary condition for the operator λ_P to be a supra closure operator is that P is closed in τ. Therefore, P cannot be considered equal to X.*

From Corollary 2.5, we have the following corollary.

Corollary 2.14. *Let (X, τ) is a topological space with a stack $\mathcal{S}$ on it. Then the collection $\tau \cup \{A \cup P : A \in \tau - \{\emptyset\},\ cl(X \setminus A) \notin \mathcal{S}\}$ is a supra topology on X^*, where $X^* = X \cup P$ for $P \cap X = \emptyset$ and $\lambda_P(X) = X^*$.*

The above corollary suggests the following definition;

Definition 2.15. *We call a supra topological space Y is a supra-extension of a (supra)topological space X if Y contains X as a supra-dense subspace. Also, if Y is a supra-extension of X then we call the supra-subspace $Y \setminus X$ of Y, the supra-remainder of Y.*

***Last point**: In a paper entitled " Embedding topological spaces in a type of generalized topological spaces ", the author has proposed a method to embedd a topological space in strong generalized topological spaces. As the definition of strong generalized topology presented by Á. Császár in [3] corresponds to the definition of supra topology introduced by Mashhour et al. in [7], so according to the adaptation of these concepts, in fact the proposed method is a way to embedd topological spaces in supra topological spaces, see [10].*

Also in [11], the author has presented a method based on extension of classical topological derived set operator to construct strong generalized spaces from topological spaces, which can be considered a way to build supra spaces from classical topological spaces.

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