AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 2(1) (2021) 73-80
DOI: 10.22060/ajmc.2021.19522.1048

On the reversible geodesics of a Finsler space endowed with a special deformed (α, β)-metric

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ABSTRACT: The scope of this paper is twofold. On the one hand, we will investigate the reversible geodesics of a Finsler space endowed with the deformed newly introduced (α, β)-metric

$$F_\epsilon(\alpha, \beta) = \frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta \tag{1}$$

where ϵ is a real parameter with $|\epsilon| < 2\sqrt{a+1}$ and $a \in \left(\frac{1}{4}, +\infty\right)$; and on the other hand, we will investigate the T-tensor for this metric.

Review History:

Received:18 January 2021
Revised:27 January 2021
Accepted:27 January 202
Available Online:01 February 2021

Keywords:

Finsler (α, β)-metric;
Deformation of an (α, β)-metric
T-Tensor

AMS Subject classifications:

53B4, 53C60

1. Introduction

Recently in paper [10], we have introduced the new deformed (α, β)-metric mentioned in the bove equation (1). This metric was obtained by deformation by the previous (α, β)-metric introduced and investigated by us in some previous papers [8], [9], [10]:

$$F = \beta + \frac{a\alpha^2 + \beta^2}{\alpha} \tag{2}$$

where $a \in \left(\frac{1}{4}, +\infty\right)$. The study of Finsler metrics with reversible geodesics represent a special field of research not only in Finsler geometry but also in physics. As we know, a Finsler space is said to have reversible geodesics if for every oriented geodesic paths, the same path traversed in the opposite direction is also a geodesic.

In a series of recent papers, [1], [6], [11], the study of Finsler spaces with reversible geodesics are discussed. Motivated by the above mentioned studies, in this paper we want to investigate the study of Finsler spaces endoweed with the metric (1), to be with reversible geodesics. In this respect, in Section 3 we will do this investigation.

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On the other hand, we know that the T-tensor plays an important role not only in Finsler geometry but also in general relativity. This tensor was introduced by Matsumoto in [7].
We will find the T-tensor associated with the metric (1) also in Section 3. This will be another main result of our paper.

2. Preliminaries

Let M be an n-dimensional, real, differentiable manifold and $\pi : TM \longrightarrow M$ be the tangent bundle of M.

Definition 2.1. *Let $F : TM - \{0\} \longrightarrow \mathbb{R}$, be a Finsler metric. The Finsler space $F^n = (M, F(x, y))$, is endowed with an (α, β)-metric, if the fundamental function F, can be written as follows: $F(x, y) = \tilde{F}(\alpha(x, y), \beta(x, y))$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ with a_{ij} a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form field on TM.*

Some interesting results in Finsler geometry, were obtained recently also in the following papers: [2], [3], [4], [13].
Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[F^2(y + su + tv)\right] |_{s,t=0}, u, v \in T_xM.$$

Let us first observe the following:

Remark 2.2. *The metric F_ϵ is also an (α, β)-metric since it can be rewritten in the following form:*

$$F_\epsilon(\alpha, \beta) = \alpha\phi(s)$$

where

$$\phi(s) = s^2 + \epsilon s + a + 1 \tag{3}$$

Also, it is easy to compute

$$\phi'(s) = 2s + \epsilon, \quad \phi''(s) = 2. \tag{4}$$

From [11], we know the following:

Definition 2.3. *[11] A special vector field Γ, on the slit tangent bundle $\tilde{TM} = TM - \{0\}$, is said to be spray on the smooth manifold M, if Γ stisfies the following properties:*
(a) In standard coordinate system (x, y) on TM, Γ can be expressed as $\Gamma = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ where $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, $\forall \lambda > 0$, G^i are called spray coefficients;
(b) G^i are smooth at $(x, y \neq 0) \in \tilde{TM}$.

As we know, locally a smooth curve $\gamma : [0, 1] \to M$ is a constant speed geodesic of (M, F) if and only if it satisfies the system of differential equations:

$$\ddot{\gamma}(t) + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0$$

with $i = 1, 2, ..., n$, where $\dot{\gamma} = \dot{\gamma}^i \frac{\partial}{\partial x^i}$. Also we recall that the Euler-Lagrange equation of (M, F) in terms of geodesic spray Γ is given by:

$$\Gamma\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial y^i} = 0.$$

Definition 2.4. *[11] If F and $\bar{F}$ are two different Finsler functions on the same differentiable manifold M, then F and $\bar{F}$ are called projectively equivalent if their geodesics coincide on a set of points.*

Lemma 2.5. *A Finsler structure (M, F) is with reversible geodesics if and only if F and its reversible function $\bar{F}$ are projectively equivalent, i.e., the geodesics of F and $\bar{F}$ coincides on a set of points.*

If we will consider the reverse geodesic spray and we will denote it by $\bar{\Gamma}$, then according to the previous lemma, F is with reversible geodesics if and only if

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial y^i} = 0 \tag{5}$$

Another well known facts from [11], is that if (M, F) is a non-Riemannian $n(n \geq 2)$-dimensional manifold, then F is with reversible geodesics if and only if $F(\alpha, \beta) = F_0(\alpha, \beta) + \epsilon\beta$, where F_0 is absolute homogeneous (α, β)-metric, ϵ is a non-zero constant and β is a closed 1-form on the manifold M.
For our metric (1), we can observe the following interesting facts: it can be rewritted as follows:

$$F_\epsilon(\alpha, \beta) = \frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta = \bar{F}(\alpha, \beta) + \epsilon\beta \tag{6}$$

where $\bar{F}(\alpha, \beta) = \frac{\beta^2+\alpha^2(a+1)}{\alpha}$, with $|\epsilon| < 2\sqrt{a+1}$ and we need to choose $\epsilon \neq 0$, so $\epsilon \in \left(-2\sqrt{a+1}, 0\right) \cup \left(0, 2\sqrt{a+1}\right)$. Therefore, we will denote: $F_\epsilon = \bar{F}(\alpha, \beta) + \epsilon\beta$ with $\bar{F}$ defined above and with $a \in \left(\frac{1}{4}, +\infty\right)$.
If Γ is the geodesic spray of F_ϵ, then $\bar{\Gamma}$ is the geodesic spray of $\bar{F}$. On the view of [11], can be obtained the following link between Γ and $\bar{\Gamma}$:

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = F_\beta\left[\bar{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i}\right].$$

because, as we know:

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = \bar{\Gamma}\left(F_\alpha \frac{\partial F}{\partial y^i} + F_\beta \frac{\partial \beta}{\partial y^i}\right) - F_\alpha \frac{\partial \alpha}{\partial x^i} - F_\beta \frac{\partial \beta}{\partial x^i} =$$

$$\bar{\Gamma}(F_\alpha)\frac{\partial \alpha}{\partial y^i} + F_\alpha\left[\bar{\Gamma}\left(\frac{\partial \alpha}{\partial y^i}\right) - \frac{\partial \alpha}{\partial x^i}\right] + \bar{\Gamma}(F_\beta)\frac{\partial \beta}{\partial y^i} + F_\beta\left[\bar{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i}\right]$$

Now, let us recall some well known results regarding the theory of T-tensor in Finsler geometry. In this respect, we will follow the paper [5]:

Lemma 2.6. *[5] The components $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ of the Cartan tensor of an (α, β)-metric are given by:*

$$C_{ijk} = \frac{\rho_1}{2\alpha}\left(h_{ij}m_k + h_{jk}m_i + h_{ik}m_j\right) + \frac{\rho_0'}{2\alpha}m_i m_j m_k,$$

where $h_{ij} = a_{ij} - \alpha_i\alpha_j$, $m_i = b_i - s\alpha_i$.

Theorem 2.7. *[5] The T-tensor of an (α, β)-metric, takes the form: $T_{ijk} = \Phi\left(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij}\right) + \Psi(h_{hk}m_i m_j + h_{hj}m_i m_k + h_{hi}m_j m_k + h_{ij}m_h m_k + h_{jk}m_i m_h + h_{ik}m_j m_h) + \Omega m_h m_i m_j m_k$,*
where:

$$\Phi = -\frac{\rho_1\phi}{2\alpha}(s + \alpha K_1 m^2)$$

$$\Psi = \frac{\rho_1\phi'}{\alpha} - \frac{\rho_1^2\phi}{\alpha\rho} - \frac{s\rho_0'\phi}{2\alpha} - \frac{\rho_1\phi m^2 K_2}{2}$$

$$\Omega = \frac{\rho_0''\phi}{2\alpha} + \frac{2\rho_0'\phi'}{\alpha} - 3\phi\left(K_2\left(\rho_1 + \frac{\rho_0' m^2}{2}\right) + \frac{\rho_1\rho_0'}{2\alpha\rho}\right).$$

$$K_1 = \frac{\rho_1}{2\alpha(\rho + m^2\phi\phi'')}$$

$$K_2 = \frac{\rho\rho_0' - 2\rho_1\phi\phi''}{2\alpha\rho(\rho + m^2\phi\phi'')}$$

here, we know that for an (α, β)-metric $F = \alpha\phi(s)$, the components $g_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}F^2$, the fundamental tensor can be calculated by the formula:

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j \tag{7}$$

where $\alpha_i = \frac{\partial \alpha}{\partial y^i}$ and

$$\rho = \phi^2 - s\phi\phi'$$

$$\rho_0 = \phi'^2 + \phi\phi''$$

$$\rho_1 = \phi\phi' - s(\phi'^2 + \phi\phi'')$$

$$\rho_2 = s^2(\phi'^2 + \phi\phi'') - s\phi\phi',$$

where $b^i = a^{ij}b_j$. These results and also other important results on this topic can be found in [12].

Proposition 2.8. *For an (α,β)-metric $F=\alpha\phi(s)$, the inverse (g^{ij}), of the metric (g_{ij}), is given by:*

$$g^{ij}=\frac{1}{\rho}a^{ij}+\mu_0 b^i b^j+\mu_1(b^i\alpha^j+b^j\alpha^i)+\mu_2\alpha^i\alpha^j \tag{8}$$

where:

$$\mu_0=-\frac{\phi\phi''}{\rho(\rho+\phi\phi''m^2)};\mu_1=-\frac{\rho_1}{\rho(\rho+\phi\phi''m^2)};$$

$$\mu_2=\frac{\rho_1\left(s\rho+(\rho_1+s\phi\phi'')m^2\right)}{\rho^2(\rho+\phi\phi''m^2)}$$

and $m^2=b^2-s^2$.

As we know from [10], we have the following result:

Theorem 2.9. *The function $F_\epsilon=\alpha\phi(s)$ is a Finsler function as defined in (1) if and only if $|s|<\min\{\sqrt{a+1},\sqrt{\frac{2b^2+a+1}{3}}\}$.*

3. Main Results

Now, we are ready to formulate our main first result:

Proposition 3.1. *Let us consider the following (α,β)-metric:*

$$F=F_\epsilon(\alpha,\beta)+\beta \tag{9}$$

where $F_\sigma(\alpha,\beta)=\frac{\beta^2+\alpha^2(a+1)}{\alpha}+\epsilon\beta$, with $a\in\left(\frac{1}{4},+\infty\right)$ and $\epsilon\in(-\sqrt{2a+1},0)\cup(0,\sqrt{2a+1})$, then F is with reversible with geodesics if and only if β is a closed 1-form.

Proof. We will start with the condition for the (α,β)-metric F, to be with reversible geodesics:

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right)-\frac{\partial F}{\partial x^i}=0$$

where $\bar{\Gamma}$ is the reverse of Γ, the geodesic spray of the metric

$$\bar{F}(\alpha,\beta)=\frac{\beta^2+\alpha^2(a+1)}{\alpha}+(\epsilon-1)\beta$$

Next, we get after easy computations:

$$\bar{\Gamma}\left(\frac{\partial F_\epsilon}{\partial y^i}\right)-\frac{\partial F}{\partial x^i}=(\bar{F})_\beta\left[\bar{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right)-\frac{\partial \beta}{\partial x^i}\right]$$

So, we conclude that:

$$\bar{\Gamma}\left(\frac{\partial F_\epsilon}{\partial y^i}\right)-\frac{\partial F}{\partial x^i}=\left(\frac{2\beta}{\alpha}+\epsilon-1\right)\left(\frac{\partial b_i}{\partial x^j}-\frac{\partial b_j}{\partial x^i}\right)y^i.$$

But $\frac{2\beta}{\alpha}+\epsilon-1=0$ is not possible, so F_ϵ is with reversible geodesics, if and only if $\left(\frac{\partial b_i}{\partial x^j}-\frac{\partial b_j}{\partial x^i}\right)=0$, i.e. F_ϵ is with reversible geodesic if and only if the differentiable 1-form β is closed in M.

□

Remark 3.2. *It is easy to observe that the (α,β)-metric F is projectively flat if and only if F_ϵ is projectively flat*

Now, we will investigate in more details the above (α,β)-metric $F_\epsilon=\frac{\beta^2+\alpha^2(a+1)}{\alpha}+\epsilon\beta$, where $a\in\left(\frac{1}{4},+\infty\right)$ and $\epsilon\in(-\sqrt{2a+1},0)\cup(0,\sqrt{2a+1})$.
Here we can observe:

$$F_\epsilon(\alpha,\beta)=\alpha\left(\frac{\beta^2}{\alpha^2}+a+1+\epsilon\frac{\beta}{\alpha}\right)=\alpha(s^2+\epsilon s+a+1).$$

So:

$$\phi(s)=s^2+\epsilon s+a+1 \tag{10}$$

with $a \in \left(\frac{1}{4}, +\infty\right)$ and $\epsilon \in (-\sqrt{2a+1}, 0) \cup (0, \sqrt{2a+1})$.
After computations, we can find the coefficients for the tensor (g_{ij}) and respectivelly for the tensor (g^{ij}) associated with the metric F_ϵ with $\phi(s) = s^2 + \epsilon s + a + 1$, as follows:

$$\rho = -s^4 - s^3\epsilon + \epsilon\, sa + \epsilon\, s + a^2 + 2\,a + 1 \tag{11}$$

$$\rho_0 = 6\,s^2 + 6\,\epsilon\, s + \epsilon^2 + 2\,a + 2 \tag{12}$$

$$\rho_1 = -4\,s^3 - 3\,s^2\epsilon + a\epsilon + \epsilon \tag{13}$$

$$\rho_2 = 4\,s^4 + 3\,s^3\epsilon - \epsilon\, sa - \epsilon\, s \tag{14}$$

$$\mu_0 = -\frac{1}{\left(s^2 + \epsilon\, s + a + 1\right)\left(\frac{1}{2} + m^2\right)} \tag{15}$$

$$\mu_1 = \frac{\left(6\,s^2 - 2\,a - 2\right)\epsilon + 8\,s^3}{\left(s^2 + \epsilon\, s + a + 1\right)^2\left(1 + 2\,m^2\right)} \tag{16}$$

$$\mu_2 = \frac{2\left(-4\,s^3 - 3\,s^2\epsilon + a\epsilon + \epsilon\right)\left(s^3 + s^2\epsilon + as + s - 2\,m^2s^3 - m^2s^2\epsilon + 2\,m^2as + m^2a\epsilon + 2\,m^2s + m^2\epsilon\right)}{\left(s^2 + \epsilon\, s + a + 1\right)^3\left(1 + 2\,m^2\right)} \tag{17}$$

Remark 3.3. *The fundamental tensors for the the metric g_{ij} and respectivelly g^{ij}, can be writted acording to (7) and (8) as follows:*

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j;$$

$$g^{ij} = \frac{1}{\rho}a^{ij} + \mu_0 b^i b^j + \mu_1(b^i\alpha^j + b^j\alpha^i) + \mu_2\alpha^i\alpha^j,$$

where the coefficients $\rho, \rho_0, \rho_1, \rho_2$, respectively μ, μ_1, μ_2, are given in the above relations (11)-(17).

Theorem 3.4. *For the (α, β)-metric $F_\epsilon(\alpha, \beta)$ with $\phi(s) = s^2 + \epsilon\, s + a + 1$, given in (10), has the following T-tensor:*
$T_{ijk} = \Phi\,(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij}) + \Psi(h_{hk}m_im_j + h_{hj}m_im_k + h_{hi}m_jm_k + h_{ij}m_hm_k + h_{jk}m_im_h + h_{ik}m_jm_h) + \Omega m_hm_im_jm_k$,
where:

$$\Phi = -\frac{1}{2}\,\frac{\left(-4\,s^3 - 3\,s^2\epsilon + (a+1)\,\epsilon\right)\left(-s^5 - s^4\epsilon + \epsilon\left(1 + a + \frac{1}{2}\,m^2\right)s^2 + (a+1)\left(a + 1 + 2\,m^2\right)s\right)}{\left(-s^2 + a + 1 + 2\,m^2\right)\alpha}$$

$$-\frac{m^2\epsilon\,(a+1)}{\left(-s^2 + a + 1 + 2\,m^2\right)\alpha}$$

$$\Psi = 6\left(-4\,s^3 - 3\,s^2\epsilon + (a+1)\,\epsilon\right)\Bigg[\left(-4\,m^2 - 2\right)s^{11} + \left(-\frac{17}{2} - 17\,m^2\right)\epsilon\, s^{10} - 12\left(\frac{1}{2} + m^2\right)\left(1 + a + \frac{19}{8}\,\epsilon^2\right)s^9$$

$$-38\left(\frac{1}{2} + m^2\right)\epsilon\left(\frac{47}{76}\,\epsilon^2 + 1 + a\right)s^8 - 12\left(\frac{1}{2} + m^2\right)\left(\frac{19}{24}\,\epsilon^4 + \left(\frac{43}{12}\,a + \frac{43}{12}\right)\epsilon^2 + (a+1)^2\right)s^7 +$$

$$\left(\left(-\frac{3}{4} - \frac{3}{2}\,m^2\right)\epsilon^5 - \frac{39}{2}\left(\frac{1}{2} + m^2\right)(a+1)\,\epsilon^3 - 24\left(\frac{1}{2} + m^2\right)(a+1)^2\,\epsilon - 2/3\,m^2\alpha\right)s^6 +$$

$$\Bigg(\left(-2\,m^2 - 1 - a - 2m^2a\right)\epsilon^4 - 12\left(\frac{1}{2} + m^2\right)(a+1)^2\,\epsilon^2 - \frac{3}{2}\,\alpha\epsilon\, m^2 -$$

$$4\left(\frac{1}{2} + m^2\right)\left(3a + a^3 + 3a^2 + 1 - \frac{1}{12}\alpha\right)\Bigg)s^5 + \Bigg(\frac{1}{2}\left(\frac{1}{2} + m^2\right)(a+1)\,\epsilon^5 + \frac{3}{2}\left(\frac{1}{2} + m^2\right)(a+1)^2\,\epsilon^3 -$$

$$\frac{13}{12}m^2\alpha\epsilon^2 - 2\left(\frac{1}{2} + m^2\right)\left(3a + 1 + 3a^2 + a^3 - \frac{5}{12}\alpha\right)\epsilon - \frac{2}{3}m^2\left(\frac{15}{16} + a\right)\alpha\Bigg)s^4$$

$$+\Bigg(\frac{2}{3}\left(\frac{1}{2} + m^2\right)(a+1)^2\,\epsilon^4 - \frac{1}{4}\,m^2\alpha\,\epsilon^3 + 3\left(\frac{1}{2} + m^2\right)\left(1 + \frac{2}{9}\,\alpha + a^3 + 3\,a + 3a^2\right)\epsilon^2 -$$

$$\frac{2}{3}\,m^2\left(\frac{15}{16} + a\right)\alpha\,\epsilon + \frac{2}{3}\left(\frac{1}{2} + m^2\right)(a+1)\,\alpha\Bigg)s^3 +$$

$$\left(\frac{1}{2}+m^2\right)\left(\left(\left(\frac{1}{6}\alpha+\frac{9}{2}a^2+\frac{9}{2}a+\frac{3}{2}a^3+\frac{3}{2}\right)\epsilon^2+(a+1)\left(a^3+3\,a^2+3\,a+\alpha+1\right)\right)\epsilon\, s^2+\right.$$

$$\left(\frac{1}{12}m^2\alpha\,(a+1)\,\epsilon^3+\frac{1}{2}\left(\frac{1}{2}+m^2\right)(a+1)\left(a^3+3\,a^2+3\,a+1+\frac{2}{3}\alpha\right)\epsilon^2+\right.$$

$$\left.\frac{1}{6}m^2\left(a+\frac{3}{4}\right)(a+1)\,\alpha\,\epsilon+\frac{1}{3}\left(\left(\frac{1}{2}+m^2\right)a^2+\left(1+2\,m^2\right)a+\frac{1}{2}-\frac{1}{2}m^2\right)\alpha\right)s+$$

$$\left.\frac{1}{12}\left(m^2\,(a+1)^2\,\epsilon^2+\left(1-m^2+2m^2a^2+4\,m^2a+2a+a^2\right)\epsilon-\frac{1}{2}m^2a\,(a+2)\right)\alpha\right]$$

$$\left(s^2+\epsilon\, s+a+1\right)^{-2}\left(\frac{1}{2}+m^2\right)^{-1}\alpha^{-2}$$

and

$$\Omega=\frac{54s^2+54\,\epsilon\, s+6\,a+6+12\,\epsilon^2}{\alpha}-3\,\left(s^2+\epsilon\, s+a+1\right)\frac{1}{\alpha(s^2+\epsilon s+a+1)(1+2m^2)}$$

$$\Bigg(\Bigg(-2(-3s^2+a+1)s\epsilon^3+\left(26\,s^4-2\,(a+1)^2\right)\epsilon^2+\left(6+36\,s^5+(15+16\,a)\,s^3+\left(-7\,a-4\,a^2-3\right)s\right)\epsilon+16\,s^6+$$

$$(15+16\,a)\,s^4+12\,s+a^2+2\,a\Bigg)(-4\,s^3-3\,s^2\epsilon+a\epsilon+\epsilon+6\,m^2s+3\,m^2\epsilon)+3\,\frac{\left(-4\,s^3-3\,s^2\epsilon+a\epsilon+\epsilon\right)(2\,s+\epsilon)}{\alpha\,\left(s^2+\epsilon\, s+a+1\right)}\Bigg)$$

Also

$$K_1=\frac{1}{2}\,\frac{-4\,s^3-3\,s^2\epsilon+a\epsilon+\epsilon}{\alpha\,(-s^2+a+1+2\,m^2)\,(s^2+\epsilon\, s+a+1)}$$

and

$$K_2=\frac{\left(6\,s^2-2\,a-2\right)s\epsilon^3+\left(26\,s^4-2\,(a+1)^2\right)\epsilon^2+\left(6+36\,s^5+(15+16\,a)\,s^3+\left(-7\,a-4\,a^2-3\right)s\right)\epsilon}{\alpha\,\left(s^2+\epsilon\, s+a+1\right)^3(1+2\,m^2)}$$

$$+\frac{16\,s^6+(15+16\,a)\,s^4+12\,s+a^2+2\,a}{\alpha\,\left(s^2+\epsilon\, s+a+1\right)^3(1+2\,m^2)}$$

Proof. The proof of this Theorem is direct, using the above relations from Theorem 2.7 and making computations to find the Ψ,Φ,Ω and also K_1 and respectively K_2. Also, here for computations we used the program Maple 13.

□

Example 3.1. *We will give an example for the above considered metric $F_\epsilon(\alpha,\beta)$, choosing $a=1$ and respectively $\epsilon=\sqrt{2}$, because $a\in\left(\frac{1}{4},+\infty\right)$ and can be choose just fromt this interval and respectivey, ϵ, must be in $(-\sqrt{2a+1},0)\cup(0,\sqrt{2a+1})$. For this values for a and respectively ϵ, we get in this case the following metric tensor for our metric:*

$$g_{ij}=-(s^2+s\sqrt{2}+2)(s^2-2)a_{ij}+(6s^2+6s\sqrt{2}+6)b_ib_j+(-4s^3-3\sqrt{2}s^2+2\sqrt{2})(b_i\alpha_j+b_j\alpha_i)+$$

$$((3\,s^3-2\,s)\,\sqrt{2}+4\,s^4)\alpha_i\alpha_j.$$

For the T-tensor, we get in this case the following values for the Ψ,Φ,Ω and also K_1 and respectively K_2:

$$K_1(s)=\frac{1}{4}\,\frac{-4\,s^3-3\,s^2\sqrt{2}+2\,\sqrt{2}}{\left(s^2+\sqrt{2}s+2\right)\alpha\,\left(-1/2\,s^2+1+m^2\right)}$$

$$K_2(s)=\frac{\left(43\,s^3-22\,s+6+36\,s^5\right)\sqrt{2}+83\,s^4-13+16\,s^6+12\,s}{\alpha\,\left(s^2+\sqrt{2}s+2\right)^3(1+2\,m^2)}$$

$$\Phi(s)=\frac{\left(\left(-2\,s^4+\left(4+m^2\right)s^2+2\,m^2\right)\sqrt{2}-2\,s^5+8\,m^2s+8\,s\right)\left(\left(\frac{3}{4}\,s^2-\frac{1}{2}\right)\sqrt{2}+s^3\right)}{\alpha\,\left(-s^2+2+2\,m^2\right)}$$

$$\Omega(s) = \frac{54\,s^2 + 54\,s\sqrt{2} + 36}{\alpha} - (3\,s^2 + 3\,s\sqrt{2} + 6)$$

$$\Bigg(\frac{\left(\left(12s^2 - 8\right)s\sqrt{2} + 83s^4 - 13 + \left(6 + 36s^5 + 31s^3 - 14\,s\right)\sqrt{2} + 16s^6 + 12s\right)}{\alpha\left(s^2 + s\sqrt{2} + 2\right)^3(1 + 2m^2)}$$

$$\frac{\left(-4s^3 - 3\sqrt{2}s^2 + 2\sqrt{2} + 6m^2s + 3\sqrt{2}m^2\right)}{\alpha\left(s^2 + s\sqrt{2} + 2\right)^3(1 + 2m^2)} + 3\frac{\left(-4s^3 - 3\sqrt{2}s^2 + 2\sqrt{2}\right)(2s + 2)}{\alpha\left(s^2 + s\sqrt{2} + 2\right)}\Bigg)$$

$$\Psi(s) = \frac{s\left(s^2 + \sqrt{2}s + 2\right)\left(2\,s^3 + 3\,s^2\sqrt{2} + 2\sqrt{2} + 12\,s\right)}{\alpha\,(s^2 - 2)} -$$

$$-\frac{\left(4\,s^3 + 3\,s^2\sqrt{2} - 2\sqrt{2}\right)m^2\left(43\,s^3\sqrt{2} - 22\sqrt{2}s + 83\,s^4 - 13 + 6\sqrt{2} + 36\sqrt{2}s^5 + 16\,s^6 + 12\,s\right)}{4\alpha\left(s^2 + \sqrt{2}s + 2\right)^2(1/2 + m^2)}$$

4. Conclusion

In this paper, we succeed to investigate a special type of deformed (α, β)-metric type, especially if a Finsler space endowed with such a metric is with reversible geodesics. Also we investigate the T-tensor for this kind of metrics and finally we investigate a particular example of such a metric. In fact, this family of deformed (α, β)-metrics which we have investigated here in this paper could generate interesting examples of metrics for particular cases of a and ϵ and in our future works we will continue to investigate new type of metrics from this family.

Remark 4.1. *In this paper, we used the Maple 13 program for computations.*

Acknowledgments

The authors express their sincere thanks to referees for their valuable suggestions and comments.

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Please cite this article using:

Laurian-Ioan Pişcoran, Cătălin Barbu, Akram Ali, On the reversible geodesics of a Finsler space endowed with a special deformed (α, β)-metric, AUT J. Math. Comput., 2(1) (2021) 73-80
DOI: 10.22060/ajmc.2021.19522.1048