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Original Article

An extension of the Cardioid distributions on circle

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ABSTRACT: A new family of distributions on the circle is introduced which is a generalization of the Cardioid distributions. The elementary properties such as mean, variance, and the characteristic function are computed. The distribution is shown to be either unimodal or bimodal. The modes are computed. The symmetry of the distribution is characterized. The parameters are shown to be canonic (i.e. uniquely determined by the distribution). This implies that the estimation problem is well-defined. We also show that this new family is a subset of distributions whose Fourier series has degree at most 2 and study the implications of this property. Finally, we study the maximum likelihood estimation for this family.

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1. Introduction

Circular data arise in many natural phenomena. The main two categories of such data are physical directions and periodical time records. Wind direction and the direction of migrating birds are two examples of physical directions. The arrival times measured by clock and the date of specific observations (in a year) are two examples of periodical time records. Circular data are usually measured and represented by degrees or radians. Axial data are another type of data that are obtained from circular data by doubling them.

Circular distributions are important tools in analyzing and inference of circular data. One of the elementary and well known circular distributions is the Cardioid distribution. This distribution has two parameters $\mu \in [0, 2\pi)$ and ρ with $|\rho| < \frac{1}{2}$ and is denoted by $C(\mu, \rho)$ and has the density function

$$f(\theta) = \frac{1}{2\pi}(1 + 2\rho \cos(\theta - \mu)).$$

Its distribution is symmetric around μ and is unimodal with a mode at μ and an anti-mode at $\mu + \pi$. This family of distributions is closed under convolution (summation of independent instances) and mixtures. For further properties see [3], page 45.

Another popular and useful circular distribution is Von Mises distribution. This distribution has two parameters $\mu \in [0, 2\pi)$ and $\kappa > 0$ and is denoted by $VM(\mu, \kappa)$ and has the probability density function

$$g(\theta) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \mu)},$$

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where

$$I_0(\kappa) = \frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos(\theta)}d\theta.$$

The parameter μ is the mean direction and κ is called the concentration parameter. This distribution is unimodal and symmetric about $\theta = \mu$. The mode of this distribution is μ and its anti-mode is $\mu + \pi$. For further information on Von Mises distributions see [3], page 36.

We should also mention a recently introduced extension of the von Mises distribution by Kato and Jones in [1]. This family is obtained by applying a Möbius transformation on the von Mises distribution. A well known member of this family is the wrapped Cauchy distribution.

The Cardioid and Von Mises families lack the necessary flexibility to capture the various behaviours of real data sets (for instance, they are always unimodal). On the other hand, the Kato-Jones family has a complicated density function which makes it hard to deal with analytically.

In this article, we introduce a new family of circular distributions which is more flexible than Cardioid and Von Mises families and has simpler parametric form than the Kato-Jones family. This new family generalizes the cardioid distributions. We call this new distribution the quadratic cardioid distribution because it's density function has a second-order triangular representation.

In the next section, after introducing the distribution, we will calculate some elementary statistics and the characteristic function. We will also discuss the mixtures and convolutions of such distributions. Finally, we will propose a maximum likelihood estimation method and apply it on a data set and will compare the results with classical methods.

2. Mathematical Definition and Properties

Let $\mu_1 \leq \mu_2 \in [0, 2\pi)$ and $r_1, r_2 \geq 0$. By a quadratic cardioid distribution, denoted by $QC(\mu_1, \mu_2, r_1, r_2)$, we mean a distribution with probability density function

$$f(\theta;\mu_1,\mu_2,r_1,r_2) = \frac{1}{I(r_1,r_2)}\Big(1 + r_1^2 + r_2^2 + 2r_1\cos(\theta-\mu_1) + 2r_2\cos(\theta-\mu_2) + 2r_1r_2\cos(2\theta-\mu_1-\mu_2)\Big)$$

where $I(r_1, r_2) = 2\pi(1 + r_1^2 + r_2^2)$.

Another representation for f is as follows,

$$f(\theta;\mu_1,\mu_2,r_1,r_2) = \frac{1}{I(r_1,r_2)}\left|1 + r_1e^{i(\theta-\mu_1)} + r_2e^{-i(\theta-\mu_2)}\right|^2$$

which makes it clear that $f \geq 0$.

The following proposition follows by substitution,

Proposition 1. *If $r_1 = r_2 = 0$ this would be the uniform distribution on the circle. If $r_2 = 0$ but $r_1 \neq 0$, the distribution becomes ordinary cardioid distribution $C(\mu_1, \frac{r_1}{1+r_1^2})$. The case $r_1 = 0$ is similar.*

Another interesting special case is when $\mu_1 = \mu_2 = \mu$. In this case the density function takes the form

$$f(\theta;\mu,r_1,r_2) = \frac{1}{I(r_1,r_2)}\Big(1 + r_1^2 + r_2^2 + 2(r_1+r_2)\cos(\theta-\mu) + 2r_1r_2\cos(2\theta-2\mu)\Big).$$

Remark 1. *For small values of r_1, r_2, the QC distribution is an approximation of the generalised Von Mises distribution which is introduced and studied in [4].*

Expectation of the QC distribution is $\frac{2\pi}{I}(r_1e^{i\mu_1}+r_2e^{i\mu_2})$ and hence the mean direction is $\bar{\theta} = Arg(r_1e^{i\mu_1}+r_2e^{i\mu_2})$. Another easy calculation gives rise to the mean resultant length, $\bar{R} = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\mu_1-\mu_2)}$.

The median of a circular distribution is defined to be a $\phi \in [0, 2\pi)$ such that $\int_\phi^{\phi+\pi} f(\theta)d\theta = 0$. For the QC distribution, we obtain,

$$0 = \int_\phi^{\phi+\pi} f(\theta)d\theta = \frac{1}{2} + 4r_1\sin(\phi-\mu_1) + 4r_2\sin(\phi-\mu_2),$$

which is equivalent to $r_1\sin(\phi-\mu_1) + r_2\sin(\phi-\mu_2) = 0$. Solving this equation gives,

$$\sin(\phi-\mu_1) = \pm\frac{r_2\sin(\mu_2-\mu_1)}{\bar{R}^2},$$

$$\sin(\phi - \mu_2) = \pm \frac{r_1 \sin(\mu_1 - \mu_2)}{\bar{R}^2}.$$

Above equations lead to two solutions which differ by π, the median is the one with $r_1 \cos(\phi-\mu_1)+r_2\cos(\phi-\mu_2) \le 0$.

The characteristic function (or Fourier series) of a circular distribution μ is defined as

$$\hat{\mu}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \mu(d\theta), \quad (n \in \mathbb{Z}).$$

A simple calculation shows that,

Proposition 2. *The characteristic function of $QC(\mu_1, \mu_2, r_2, r_2)$ is*

$$\hat{\mu}(n) = \begin{cases} \frac{1}{2\pi} & n = 0 \\ (r_1 e^{-i\mu_1} + r_2 e^{-i\mu_2})/I & n = 1 \\ (r_1 e^{i\mu_1} + r_2 e^{i\mu_2})/I & n = -1 \\ r_1 r_2 e^{-i(\mu_1+\mu_2)}/I & n = 2 \\ r_1 r_2 e^{i(\mu_1+\mu_2)}/I & n = -2 \\ 0 & n \neq 0, \pm 1, \pm 2 \end{cases}$$

3. The Shape of the Distribution

In this section, we determine that for what values of the parameters, the QC distribution can be symmetric or asymmetric, unimodal or bimodal.

Proposition 3. *The QC distribution is symmetric if and only if at least one of the following holds:*

(i) $r_1 = r_2$.

(ii) $\mu_1 = \mu_2$.

(iii) *at least one of r_1 and r_2 are zero.*

In case (i), the distribution is symmetric about $\frac{\mu_1+\mu_2}{2}$, in case (ii), the distribution is symmetric about $\mu_1 = \mu_2$ and in case (iii), the distribution is symmetric about μ_1 (or μ_2).

Proof. The if part is straightforward. For the only if part, assume the distribution is symmetric about θ_0 (and hence about $\theta_0 + \pi$). Denote the density function by $f(\theta)$ and assume that both r_1 and r_2 are nonzero. We have,

$$f^{(2n)}(\theta) = \frac{1}{I(r_1, r_2)} (-1)^n (2r_1 \cos(\theta - \mu_1) + 2r_2 \cos(\theta - \mu_2) + 2^{2n+1} r_1 r_2 \cos(2\theta - \mu_1 - \mu_2)).$$

Hence we have,

$$\lim_{n\to\infty} f^{(2n)}(\theta) 2^{-2n} = r_1 r_2 \cos(2\theta - \mu_1 - \mu_2).$$

Since f is symmetric about θ_0 then so is $f^{(2n)}$ and hence $\cos(2\theta - \mu_1 - \mu_2)$. This implies that $\theta_0 = \mu_1 + \mu_2$ (or $\mu_1 + \mu_2 + \pi$ which makes no difference in the remainder of the proof). And also we find that $2r_1 \cos(\theta - \mu_1) + 2r_2 \cos(\theta - \mu_2)$ is also symmetric about θ_0. This, combined with $\mu_1 = \mu_2$ implies that $r_1 = r_2$.

□

Modes and anti-modes of a circular distribution correspond to the local maxima and minima of its density function. The following proposition characterizes the modes and anti-modes for QC distributions.

Proposition 4. (i) *In general, modes and anti-modes are the roots of the following function,*

$$r_1 \sin(\theta - \mu_1) + r_2 \sin(\theta - \mu_2) + 2r_1 r_2 \sin(2\theta - \mu_1 - \mu_2). \tag{1}$$

(ii) *The QC distribution has either 1 mode and 1 anti-mode or 2 modes and 2 antimodes.*

Proof. (i) Follows easily from calculation of $f'(\theta)$.

(ii) It is obvious that a periodic function has at least one maximum and one minimum in its period. Hence at least one mode and at least one anti-mode exist. On the other hand equation (1) can be turned in to a 4th degree equation by expanding the expression in terms of $t = \tan(\frac{\theta}{2})$. Hence at most 4 roots exist and noting that the number of nodes and anti-nodes should be equal, the statement follows.

□

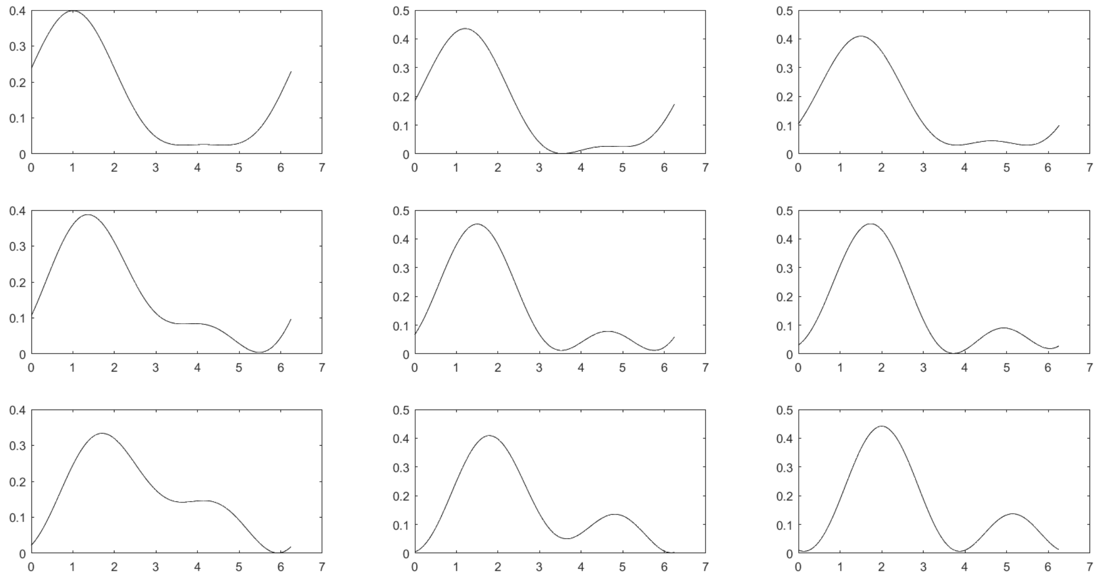
Corollary 3.1. *In the special case $\mu_1 = \mu_2 = \mu$, two solutions of (1) are $\theta = \mu$ and $\theta = \mu + \pi$. The former is always a mode and the latter is a mode if and only if $4r_1r_2 > r_1 + r_2$. If this inequality holds then there are two other solutions for (1) which are the solutions of $\cos(\theta - \mu_1) = -\frac{r_1+r_2}{4r_1r_2}$ and both are anti-modes.*

Remark 2. *In general, the modes and anti-modes can be calculated analytically using Ferrari's method for the 4th degree equation mentioned in the proof of proposition 4.*

Figure 1 shows the different shapes of the density of a quadratic Cardioid distribution for different values of parameters.

Columns from left to right correspond respectively to $r_1 = \frac{1}{2}, 1, \frac{3}{2}$ and $\mu_1 = 1, \frac{3}{2}, 2$ and rows from top to bottom correspond respectively to $r_2 = \frac{1}{2}, 1, \frac{3}{2}$ and $\mu_2 = \frac{1}{2}, 1, \frac{3}{2}$.

Figure 1: graphs of densities of the Q-Cardioid distribution for different values of parameters



4. QC as a parametrization of second order positive definite Fourier series

As we showed in proposition 2, the Fourier series of the QC distribution has degree 2 (i.e coefficients of degrees higher than 2 are zero). In this section we will further investigate the properties of the QC distribution from the Fourier series point of view.

Let $\mathcal{M}$, $\mathcal{M}^+$ and $\mathcal{M}^\pi$ be respectively the set of all signed Borel measures, positive Borel measures and probability Borel measures on $\mathbb{S}^1$.

We define the spaces $\mathcal{T}_N$ to be the set of all positive signed Borel measures on $\mathbb{S}^1$ whose Fourier series has degree at most N. In other words,

$$\mathcal{T}_N = \{\mu \in \mathcal{M} : \hat{\mu}(n) = 0 \quad \forall n \notin \{-N, \dots, 0, \dots, N\}\},$$

and define

$$\mathcal{T}_N^+ = \mathcal{T}_N \cap \mathcal{M}^+, \quad \mathcal{T}_N^\pi = \mathcal{T}_N \cap \mathcal{M}^\pi.$$

Proposition 5. *$\mathcal{T}_N^+$ is closed under linear combination and convolution. $\mathcal{T}_N^\pi$ is closed under mixture and convolution.*

Proof. Since Fourier transform is a linear transformation, hence closedness under linear combinations is trivial. On the other hand, convolution of measures translates into multiplication of Fourier transforms and hence closedness under convolution also follows. The statement for $\mathcal{T}_N^\pi$ follows easily from the fact that $\mathcal{M}^\pi$ is closed under mixture and convolution. □

Let $\mathcal{C}$ and $\mathcal{QC}$ be respectively the space of all Cardioid and Quadratic-Cardioid distributions.

The following proposition shows the relation between Cardioid distributions and $\mathcal{T}_N^\pi$ spaces.

Proposition 6. **(i)** $\mathcal{C} = \mathcal{T}_1^\pi$.

(ii) $\mathcal{QC} \subset \mathcal{T}_2^\pi$.

Proof. **(i)** Let f be the density function of some $\mu \in \mathcal{T}_1^\pi$. We have

$$f(\theta) = \hat{f}(0) + \hat{f}(1)e^{i\theta} + \hat{f}(-1)e^{-i\theta}.$$

Since f is real we can replace the above expression with its real part. For suitable constants a_0, a_1 and b_1 we obtain,

$$f(\theta) = a_0 + a_1\cos(\theta) + b_1\sin(\theta) = a_0 + c\cos(\theta - \mu),$$

where $c = \sqrt{a_1^2 + b_1^2}$ and $\mu = \arctan(a_1/b_1)$. Integrating over $[0, 2\pi]$ implies $a_0 = \frac{1}{2\pi}$ and positivity of f implies that $|c| < \frac{1}{\pi}$ which implies the statement.

(ii) Follows obviously from proposition 2.

□

Remark 3. *Indeed it may be the case that $\mathcal{QC} = \mathcal{T}_2^\pi$ but we have not yet succeeded in proving or disproving it.*

The previous proposition implies that the QC distributions are actually parametrizing the space $\mathcal{T}_2^\pi$ at least partially. A good question is that to what extent is this parametrization complete? In other words, what portion of $\mathcal{T}_2^\pi$ is covered by this parametrization?

To answer this question we need to characterize the elements of $\mathcal{T}_2^\pi$ in terms of their Fourier series. We use a well-known theorem due to Bochner which provides a necessary and sufficient condition for a function to be the characteristic function of a probability distribution. The Bochner theorem holds in general for all probability measures on dual group of any Abelian group, but we state it here only for the special case of probability measures on $\mathbb{S}^1$.

Theorem 4.1 (Bochner). *A function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ is the Fourier series of a positive probability measure on $\mathbb{S}^1$ if and only if $\hat{f}(0) = \frac{1}{2\pi}$ and the following matrix is positive definite:*

$$\begin{bmatrix} \hat{f}(0) & \hat{f}(1) & \hat{f}(2) & \cdots & \cdots \\ \hat{f}(-1) & \hat{f}(0) & \hat{f}(1) & \hat{f}(2) & \cdots \\ \hat{f}(-2) & \hat{f}(-1) & \hat{f}(0) & \hat{f}(1) & \ddots \\ \vdots & \hat{f}(-2) & \hat{f}(-1) & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

Let $\hat{\nu}$ be the Fourier series of an element ν in $\mathcal{T}_2^\pi$. We assume $\hat{\nu}(1) = \frac{c_1}{2\pi}$ and $\hat{\nu}(2) = \frac{c_2}{2\pi}$. It follows that $\hat{\nu}(-1) = \frac{\bar{c}_1}{2\pi}$ and $\hat{\nu}(-2) = \frac{\bar{c}_2}{2\pi}$.

Applying the Bochner theorem now implies,

Proposition 7. *$\nu \in \mathcal{T}_2^\pi$ if and only if $\hat{\nu}(0) = 2\pi$, $\hat{\nu}(1) = 2\pi c_1$ and $\hat{\nu}(2) = 2\pi c_2$ and the following pentadiagonal matrix is positive definite:*

$$\begin{bmatrix} 1 & c_1 & c_2 & 0 & \cdots \\ \bar{c}_1 & 1 & c_1 & c_2 & \cdots \\ \bar{c}_2 & \bar{c}_1 & 1 & c_1 & \ddots \\ 0 & \bar{c}_2 & \bar{c}_1 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

Applying the determinant criterion for positive definiteness, gives us the following necessary conditions:

Proposition 8. *The following are necessary conditions for $\nu \in \mathcal{T}_2^\pi$,*

$$-c_1\overline{c_1} + 1 > 0,$$

$$c_1^2\overline{c_2} - 2c_1\overline{c_1} + c_2\overline{c_1}^2 - c_2\overline{c_2} + 1 > 0,$$

$$c_1^2\overline{c_1}^2 + 2c_1^2\overline{c_2} - 2c_1c_2\overline{c_1}\,\overline{c_2} - 3c_1\overline{c_1} + c_2^2\overline{c_2}^2 + 2c_2\overline{c_1}^2 - 2c_2\overline{c_2} + 1 > 0.$$

Remark 4. *One may think of generalizing the QC distributions as follows,*

$$f(\theta;\mu,r)=\frac{1}{I(r)}\Big(1+\sum_{i=1}^{p}r_i^2+\sum_{i=1}^{p}2r_i\cos(\theta-\mu_i)+\sum_{i=1}^{p}\sum_{j=1}^{p}2r_ir_j\cos(2\theta-\mu_i-\mu_j)\Big).$$

This family has $2p$ *parameters. Although taking* $p>2$ *may result in a larger family than QC, but the disadvantage is that since this new family is still a subset of* $\mathcal{T}_2^{\pi}$ *(which is a four-dimensional family), hence the above parametrization can not be invertible. This non-invertibility makes the estimation problem ill-posed.*

5. Maximum Likelihood Estimation

In this section, we investigate the estimation problem for QC distributions. First we show that the estimation problem is well-posed in the sense that the parameters are uniquely determined by the distribution. Then we will apply the maximum likelihood method to a real data set.

Theorem 5.1. *Parameters of a QC distribution are uniquely determined by the distribution.*

Proof. Assume that $QC(\mu_1,\mu_2,r_2,r_2)\approx QC(\mu_1',\mu_2',r_1',r_2')$. Define

$$g(z)=r_1e^{-i\mu_1}z^2+z+r_2e^{i\mu_2},$$

and

$$h(z)=r_1'e^{-i\mu_1'}z^2+z+r_2'e^{i\mu_2'}.$$

It follows from the assumption that for $z=e^{i\theta}$, $\left|\frac{g(z)}{h(z)}\right|$ is a constant c. Hence the function $\frac{g(z)}{ch(z)}$ maps the unit circle $|z|=1$ into itself.

Now it follows from the Schwartz lemma that $\frac{g(z)}{ch(z)}$ is a product of two Mobius functions $e^{i\alpha}\frac{z-a}{1-\bar{a}z}$ and $e^{i\beta}\frac{z-b}{1-\bar{b}z}$. Substituting implies that $(\mu_1,\mu_2,r_1,r_2)=(\mu_1',\mu_2\prime,r_1',r_2')$. □

The log-likelihood function of a data set $\{\theta_1,\dots,\theta_N\}$ is

$$ll(\mu_1,\mu_1,r_1,r_2)=\sum_{i=1}^{N}\log\Big(1+r_1^2+r_2^2+2r_1\cos(\theta_i-\mu_1)+2r_2\cos(\theta_i-\mu_2)+2r_1r_2\cos(2\theta_i-\mu_1-\mu_2)\Big)-N\log(2\pi(1+r_1^2+r_2^2)).$$

In the following, we apply the maximum likelihood estimation on a real data set. The data, is the minute-by-minute exchange volume of Bitcoin (BTC) for the time period of Jan 2012 to April 2020. The data has been downloaded from Kaggle website (www.kaggle.com/mczielinski/bitcoin-historical-data). The advantage of the Bitcoin data is that since it is a 24-hour market, the exchange data can be treated as a circular data and then one can inference on the daily or weekly activity of this market. For this research we have taken the time period to be a week.

We have set the starting parameters of optimization to be those of ordinary Cardioid distribution. Figure 2 shows the resulting fitted QC distribution.

Figure 2: Histogram of the exchange volume of BTC and the density of the fitted QC distribution

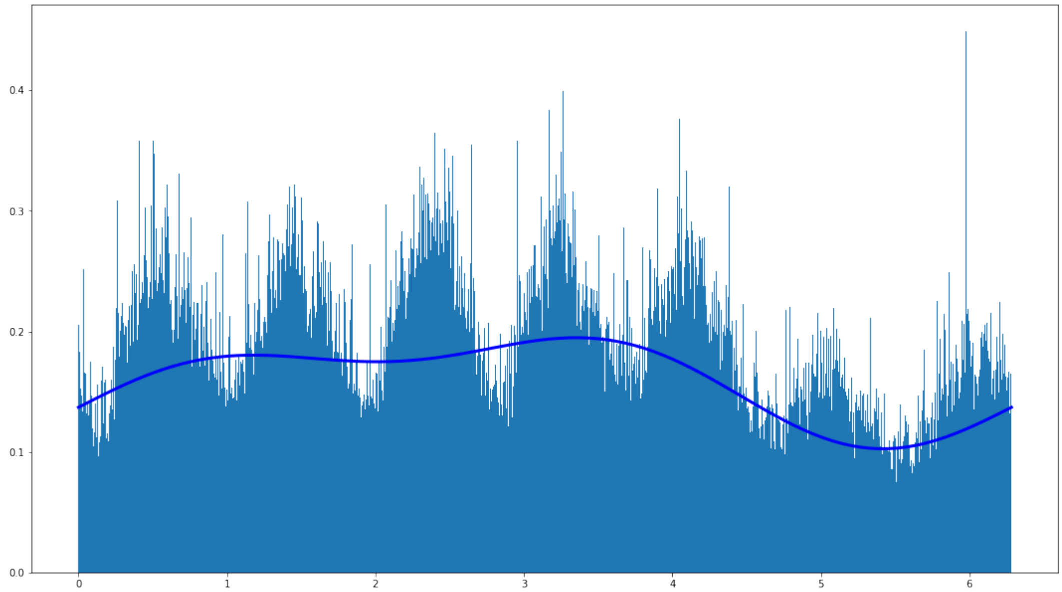


Table 1 shows the fitted parameters for three different families of circular distributions along with their log-likelihoods, AIC, and BIC. It is clear that QC distributions outperform the other two families of distributions, by any of the three criteria.

	Cardioid	von Mises	Quadratic Cardioid
Parameters	$\mu = 2.40, r = 0.12$	$\mu = 0.23, \kappa = 2.43$	$\mu_1 = 0.02, \mu_2 = 2.28, r_1 = 0.27, r_2 = 8.01$
log-Likelihood	-56185588	-56213094	-56158331
AIC	112371179	112426193	112316666
BIC	112371208	112426250	112316724

Table 1: Fitted parameters, log-Likelihood, AIC and BIC of the best fitted distribution among three different families of circular distributions

6. Conclusion

In this article we introduced a new family of circular distributions, called QC distributions, which is more flexible than well-known circular distributions and at the same time is computationally tractable. This four-parameter family is a subset of the family of circular distributions whose fourier series has second order. The descriptive statistics of QC distributions, including mean, median, modes and characteristic function has been calculated analytically. The maximum-likelihood estimation for this family has been investigated and has been implemented on a real data set. The results show that QC distributions outperform well-known distributions in the sense of likelihood, AIC and BIC.

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