AUT Journal of Mathematics and Computing

AUT J. Math. Com., 2(1) (2021) 37-44
DOI: 10.22060/ajmc.2020.18460.1036

Conservation law and Lie symmetry analysis of Foam Drainage equation

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ABSTRACT: In this paper, using the Lie group analysis method, we study the group invariant of the Foam Drainage equation. It shows that this equation can be reduced to ODE. Also we apply the Lie-group classical, and the nonclassical method due to Bluman and Cole to deduce symmetries of the Foam Drainage equation. and we prove that the nonclassical method applied to the equation leads to new reductions, which cannot be obtained by Lie classical symmetries. Also this paper shows how to construct directly the local conservation laws for this equation.

Review History:
Received:18 May 2020
Revised:10 July 2020
Accepted:13 July 2020
Available Online:01 February 2021

Keywords:
Lie group analysis
Conservation law
Optimal system
Foam Drainage equation
Nonclassical symmetry

1. Introduction

Symmetry (joined to simplicity) has been, is, and probably will continue to be, an elegant and useful tool in the formulation and exploitation of the *laws of nature*. The request of symmetry accounts for the regularities of the laws that are independent of some inessential circumstances. For instance, the reproducibility of experiments in different places at different times relies on the invariance of the lawsof nature under space translation and rotation (homogeneity and isotropy of space), and time translation (homogeneity of time). Without such regularities physical events probably would remain out of our knowledge, and the formulation of the laws themselves would be impossible. An important implication of symmetry in physics and in mathematics is the existence of conservation laws. This connection has been noticed in 1918, when Emmy Noether [13] proved her famous theorem relating continuous symmetries and conservation laws.

In the nineteenth century a great advance arose when the Norwegian mathematician Sophus Lie began to investigate the continuous groups of transformations leaving differential equations invariant, creating what is now called the *symmetry analysis of differential equations*. Thus, symmetry analysis of differential equations was developed and applied by Sophus Lie during the period 1872-1899 [10, 11]. This theory enables to derive solutions of differential equations in a completely algorithmic way without appealing to special lucky guesses.

Most scientific problems and physical phenomena occur nonlinearly. Except in a limited number of these problems, finding the exact analytical solutions of such problems are rather difficult. Therefore, there have been attempts to develop new techniques for obtaining analytical solutions which reasonably approximate the exact solutions [12]. In recent years, several such techniques have drawn special attention, such as Lie group [10, 11] ,the homogeneous balance method, Adomian's decomposition method (ADM), and etc.

Foams are of great importance in many technological processes and applications, and their properties are subject of intensive studies from both practical and scientific points of view [16]. Liquid foam is an example of soft matter

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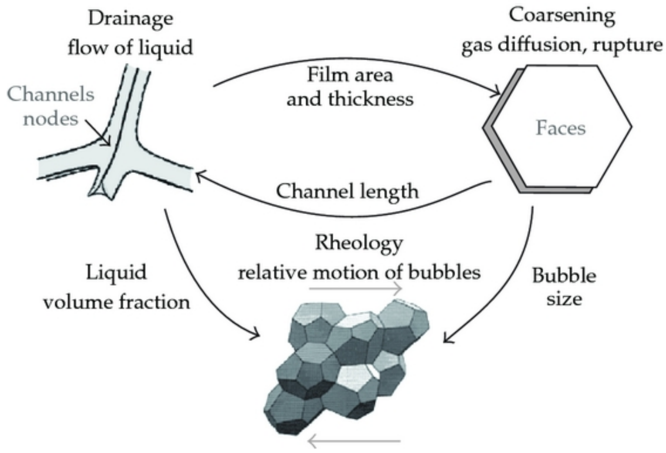

Figure 1: Schematic illustration of the interdependence of drainage, coarsening, and rheology of foams [16].

(or complex fluid) with a very welldefined structure that first clearly described by Joseph plateau in the 19th century. Weaire et al. [24] showed in their work simple answers to many such questions exist, but no going experiments continue to challenge our understanding. Foams and emulsions are wellknown to scientists and the general public alike because of their everyday occurrence [14, 23]. Foams are common in foods and personal care products such as creams and lotions, and foams often occur, even when not desired, during cleaning (clothes, dishes, scrubbing) and dispensing processes [15]. They have important applications in the food and chemical industries, firefighting, mineral processing, and structural material science [7]. Less obviously, they appearinacoustic cladding, lightweightmechanical components, andimpactabsorbingparts on cars, heat exchangers, and textured wallpapers (incorporated as foaming inks) and even have an analogy in cosmology. The packing of bubbles or cells can form both random and symmetrical arrays, such as sea foam and bees' honeycomb. History connects foams with a number of eminent scientists, and foams continue to excite imaginations [25]. There are now many applications of polymeric foams [5] and more recently metallic foams, which are foams made of metals such as aluminum [2]. Some commonly mentioned applications include the use of foams for reducing the impact of explosions and for cleaning up oil spills. In addition, industrial applications of polymeric foams and porous metals include their use for structural purposes and as heat exchange media analogous to common "finned" structures [8]. Polymeric foams are used in cushions and packing and structural materials [6]. Glass, ceramic, and metal foams [1] can also be made and find an increasing number of new applications. In addition, mineral processing utilizes foam to separate valuable products by flotation. Finally, foams enter geophysical studies of the mechanics of volcanic eruptions [15]. Recent research in foams and emulsions has centered on three topics which are often treated separately but are, in fact, interdependent: drainage, coarsening, and rheology; see Figure 1. We focus here on a quantitative description of the coupling of drainage and coarsening. Foam drainage is the flow of liquid through channels (plateau borders) and nodes (intersections of four channels) between the bubbles, driven by gravity and capillarity [9]. During foam production, the material is in the liquid state, and fluid can rearrange while the bubble structure stays relatively unchanged. The flow of liquid relative to the bubbles is called drainage. Generally, drainage is driven by gravity and/or capillary (surface tension) forces and is resisted by viscous forces [15]. Because of their limited time stability and despite the numerous studies reported in the literature, many of their properties are still not well understood, in particular the drainage of the liquid in between the bubbles under the influence of gravity [22, 3]. Drainage plays an important role in foam stability. Indeed, when foam dries, its structure becomes more fragile; the liquid films between adjacent bubbles being thinner, then can break, leading to foam collapse. In the case of aqueous foams, surfactant is added into water, and it adsorbs at the surface of the films, protecting them against rupture [4]. Most of the basic rules that explain the stability of liquid gas foams were introduced over 100 years ago by the Belgian Joseph Plateau who was blind before he completed his important book on the subject. This modern-day book by Weaire and Hutzler provides valuable summaries of plateaus work on the laws of equilibrium of soap films, and it is especially useful since the original 1873 French text does not appear to be in a fully translated English version. Weaire and Hutzler note that Sir W. Thompson (Lord Kelvin) was simulated by Plateau's book to examine the optimum packing of free space by regular geometrical cells. His solution to the problem remained the best until quite recently. Why does this area of theoretical research, still active today, have connections with the apparently frivolous theme of bubbles? It is because the packing of free space involves the minimization of the surface energy of the structure (i.e., least amount of boundary material). Thus, one might ask why such an often-observed medium as a foam has not provided the optimum solution to this problem much earlier; perhaps, this shows that observation is often biased towards what one expects to see, rather than to the unexpected. Also, in nature, there are packing problems, such as the bees' honeycomb. Its shaped ends provide a nice example of Plateau's rules in a natural environment [25]. Recent theoretical studies by Verbist and Weaire describe the main features of both free drainage [17, 19], where liquid drains out of a foam due to gravity, and forced drainage [18], where liquid is introduced to the top of a column of foam. In the latter case, a solitary wave of constant velocity is generated when liquid is added at a constant rate [20]. Forced foam drainage may well be the best prototype for certain general phenomena described by nonlinear differential equations, particularly the type of solitary wave which is most familiar in tidal bores. The model developed by Verbist and Weaire idealizes the network of Plateau borders, through which the majority of liquid is assumed to drain, as a set of N identical pipes of cross section A, which is a function of position and time [21].

$$\frac{\partial A}{\partial t}+\frac{\partial}{\partial x}\left(A^2-\frac{\sqrt{A}}{2}\frac{\partial A}{\partial x}\right)=0, \tag{1}$$

where x and t are scaled position and time coordinates, respectively.

2. Lie Symmetry of the Foam Drainage Equation (FDE)

We consider the Foam Drainage equation. We first use the transformation $A(x,t)=u^2(x,t)$ to convert (1) to

$$\text{FDE}:\quad u_t=\frac{1}{2}uu_{xx}+u_x^2-2u^2u_x. \tag{2}$$

Lie method of infinitesimal transformation groups which essentially reduces the number of independent variables in PDE and reduces the order of ODE has been widely used in equations of mathematical physics, The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations and the associated determining equations are an over determined linear system. We let the group of infinitesimal transformations be defined as

$$\overline{x}=x+\varepsilon\xi^1(x,t,u)+O(\varepsilon^2),\qquad \overline{t}=t+\varepsilon\xi^2(x,t,u)+O(\varepsilon^2),\qquad \overline{u}=u+\varepsilon\eta(x,t,u)+O(\varepsilon^2), \tag{3}$$

and impose the condition of invariance on (1). The invariance under (3) means that if u is solution of (1), then u^* is also asolution of it.

We conside the Lie point symmetry generator $\mathbf{v}=\xi^1(x,t,u)\partial_x+\xi^2(x,t,u)\partial_t+\eta(x,t,u)\partial_u$ of Foam Drainage equation, then $\boldsymbol{v}$ must satisfy Lie's symmetry condition $Pr^{(2)}\mathbf{v}\ (\Delta)=0$, To obtain the infinitesimal generators $\mathbf{v}$, we need to determin all possible coefficient function ξ^1,ξ^2 and η so that the corresponding one-parameter group $\exp(\varepsilon\mathbf{v})$ is a symmetry group of the FDE equation, we need to know the second prolongation

$$pr^{(2)}\mathbf{v}=\mathbf{v}+\eta^x\frac{\partial}{\partial u_x}+\eta^t\frac{\partial}{\partial u_t}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xt}\frac{\partial}{\partial u_{xt}}+\eta^{tt}\frac{\partial}{\partial u_{tt}}, \tag{4}$$

of $\mathbf{v}$, with the coefficients:

$$\begin{aligned}
\eta^x&=\mathrm{D}_x\eta+u_x\mathrm{D}_u\eta-u_x\mathrm{D}_x\xi^1-{u_x}^2\mathrm{D}_u\xi^1-u_t\mathrm{D}_x\xi^2-u_tu_x\mathrm{D}_u\xi^2,\\
\eta^t&=\mathrm{D}_t\eta+u_t\mathrm{D}_u\eta-u_x\mathrm{D}_t\xi^1-u_xu_t\mathrm{D}_u\xi^1-u_t\mathrm{D}_t\xi^2-{u_t}^2\mathrm{D}_u\xi^2,\\
\eta^{xx}&=\mathrm{D}_x^2\eta+2u_x\mathrm{D}_u\mathrm{D}_x\eta+\cdots-2u_xu_{xt}\mathrm{D}_u\xi^2,\\
\eta^{xt}&=\mathrm{D}_x\mathrm{D}_t\eta+u_t\mathrm{D}_u\mathrm{D}_x\eta-\cdots-u_{tt}u_x\mathrm{D}_u\xi^2,\\
\eta^{tt}&=\mathrm{D}_t^2\eta+2u_t\mathrm{D}_u\mathrm{D}_t\eta-\cdots-2u_{tt}\mathrm{D}_t\xi^2,
\end{aligned}$$

Applying $pr^{(2)}\mathbf{v}$ to FDE equation , we find the infinitesimal generator. So that we need to solve the equations yields:

$$\begin{aligned}
&\xi^2_x=0,\quad \xi^2_u=0,\quad \xi^1_u=0,\quad 2u\xi^1_x-\eta-u\xi^2_t=0,\quad 2\eta_t+4u^2\eta_x-u\eta\eta_{xx}=0,\\
&u\eta_{uu}+2\eta_u-4\xi^1_x+2\xi^2_t=0,\qquad -4\xi^1_t+8u^2\xi^2_t-8u^2\xi^1_x-8\eta_x-4u\eta_{ux}+16u\eta+2u\xi^1_{xx}=0,
\end{aligned}$$

The solution of the above system gives the following coefficients of the vector field v:

$$\xi^1=c_1x+c_3,\quad \xi^2=3c_1t+c_2,\quad \eta=-c_1u, \tag{5}$$

Where c_1, c_2, and c_3 are arbitary constants, thus the Lie algebra $\mathfrak{g}$ of the FDE equation is spanned by the three vector fields $\mathbf{v}_1=x\partial_x+3t\partial_t-u\partial_u$, $\mathbf{v}_2=\partial_t$, and $\mathbf{v}_3=\partial_x$. Also their commutator table is

Table 1

[,]	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	$-3\mathbf{v}_2$	$-\mathbf{v}_3$
$\mathbf{v}_2$	$3\mathbf{v}_2$	0	0
$\mathbf{v}_3$	$\mathbf{v}_3$	0	0

Table 2

[,]	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2 - \varepsilon\mathbf{v}_3$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1 + \varepsilon\mathbf{v}_3$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$

3. Group Invariant Solutions

Theorem. The one-parameter groups $G_i(\varepsilon)$ generated by the v_1, v_2, v_3 are given in the following table:

$$G_1(x,t,u) = (xe^{\varepsilon}, te^{3\varepsilon}, ue^{-\varepsilon}), \qquad G_2(x,t,u) = (x, t+\varepsilon, u), \qquad G_3(x,t,u) = (\varepsilon + x, t, u), \tag{6}$$

where entries give the transformed point $exp(\varepsilon \boldsymbol{v}_i)(x,t,u) = (\overline{x}, \overline{t}, \overline{u})$, $i = 1,2,3$. ☐

Taking into account the fact that generally to each one parameter subgroups of the full symmetry group of a system, there will associate a family of solutions called invariant solutions, the following theorem can be stated:

Theorem. If $u = f(x,t)$ is a solution of Eq. (1), so are the functions

$$G_1(\varepsilon)\cdot f(x,t) = e^{\varepsilon} f(xe^{\varepsilon}, te^{3\varepsilon}), \qquad G_2(\varepsilon)\cdot f(x,t) = f(x, t+\varepsilon), \qquad G_3(\varepsilon)\cdot f(x,t) = f(x+\varepsilon, t), \tag{7}$$

where ε is a real number. ☐

Here we can find the general group of the symmetries by considering a general linear combination "$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$" of the given vector fields. In particular if $\mathfrak{g}$ is the action of the symmetry group near the identity, it can be represented in the form $\mathfrak{g} = \{\exp(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) \,:\, c_1, c_2, c_3 \in \mathbb{R}\}$.

4. Optimal system of the Benny equation

This part using the adjoint representation for classifying group-invariant solutions. let G a Lie group. An *optimal system* of subgroup is a list of conjugace nequivalent subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of subalgebras forms an *optimal system* if every subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{h} = \mathrm{Ad},\ g(h), g \in G$. We finding exact solutions and performing symmetry reductions of differential equations. As any transformation in the symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation.

The adjoint action is given by the Lie series $\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}_j = \sum_{n=0}^{\infty}(\varepsilon^n/n!)(ad\ \mathbf{v}_i)^n(\mathbf{v}_j)$, where $[\mathbf{v}_i, \mathbf{v}_j]$ is a commutator for the Lie algebra, ε is a parameter, and $i,j = 1,2,3$, and also table Adjoint with $(i,j)-$th entry indicating $\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v}_j)$: where ε is a real number.

Let $F_i^{\varepsilon} : \mathfrak{g} \to \mathfrak{g}$ defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v})$ is a linear map, for $i = 1,2,3$. The matrices M_i^{ε} of F_i^{ε}, $i = 1,2,3$ with respect to basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

$$M_1^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{3\varepsilon} & 0 \\ 0 & 0 & e^{\varepsilon} \end{pmatrix}, \qquad M_2^{\varepsilon} = \begin{pmatrix} 1 & -\varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3^{\varepsilon} = \begin{pmatrix} 1 & 0 & -\varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

by acting above matrices on a vector field $\mathbf{v}$ alternatively we can show that a one-dimensional optimal system of $\mathfrak{g}$ is given by

$$Y_1 = \mathbf{v}_1, \qquad Y_2 = \mathbf{v}_2, \qquad Y_3 = \mathbf{v}_3, \qquad Y_4 = \mathbf{v}_2 + \mathbf{v}_3, \qquad Y_5 = \mathbf{v}_2 - \mathbf{v}_3, \tag{9}$$

5. Nonclassical Symmetry of FDE equation

Bluman and Cole in 1969, proposed the nonclassical symmetry method to obtain new exact solution of the linear heat equation. Consider a kth order system Δ_1 of differential equations

$$\Delta_\nu(x, u, u^{(k)}) = 0, \quad \nu = 1, \cdots, l, \tag{10}$$

in n independent variables $x = x(x_1, \cdots, x_n)$ and q dependent variables $u = u(u^1, \cdots, u^q)$ with u^k denoting the derivatives of the u's with respect to the $x's$ up to order k. Suppose that $\mathbf{v}$ is a vector field on the space $\mathbb{R}^n \times \mathbb{R}^q$ of independent and dependent variables: $\mathbf{v} = \sum_{i=1}^n \xi^i(x, u)\, \partial_{x_i} + \sum_{\alpha=1}^q \phi^\alpha(x, u)\, \partial_{u^\alpha}$. In what follows, the derivatives ∂_{x_i}, ∂_{u^α} and so on will be for short denoted by ∂_{x_i}, $\partial_{u^\alpha}$and so on. The graph of a solution

$$u^\alpha = f^\alpha(x_1, \cdots, x_n), \quad \alpha = 1, \cdots, q, \tag{11}$$

to the system defines an n-dimensional submanifold $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}^q$ of the space of independent and dependent variables. The solution will be invariant under the one-parameter subgroup generated by $\mathbf{v}$ if and only if Γ_f is an invariant submanifold of this group. By applying the well known criterion of invariance of a submanifold under a vector field we get that (11) is invariant under $\mathbf{v}$ if and only if f satisfies the first order system Δ_2 of partial differential equations: $Q^\alpha(x, u, u^{(1)}) = \phi^\alpha(x, u) - \sum_{i=1}^n \xi^i(x, u)\, \partial u^\alpha / \partial x_i = 0$, that $\alpha = 1, \cdots, q$ known as the *invariant surfacec onditions*. The q-tuple $Q = (Q^1, \cdots, Q^q)$ is known as the characteristic of the vector field $\mathbf{v}$. Since all the solutions of (10) are invariant under $\mathbf{v}$ the first prolongation $\mathbf{v}^{(1)}$ of $\mathbf{v}$ is tangent to Δ_2. Therefore, we conclude that invariant solutions of the system (10) are infact solutions of the joint overdetermined system (10) In what follows, the k-th prolongation of the invariant surface conditions (10) will be denoted by Δ_k, which is a kth order system of partial differential equations obtained by appending to (10) its partial derivatives with respect to the independent variables of orders $j \le k-1$. For the system (10) to be compatible, the k-th prolongation $\mathbf{v}^{(k)}$ of the vector field $\mathbf{v}$ must be tangent to the intersection $\Delta \cap \Delta_k$:

$$\mathbf{v}^{(k)}(\Delta_\nu)|_{\Delta \cap \Delta_k} = 0, \quad \nu = 1, \cdots, l, \tag{12}$$

If the equations (12) are satisfied, then the vector field $\mathbf{v}$ is called a nonclassical infinitesimal symmetry of the system (10).

We define $\mathbf{v} = \tau^1\partial_x + \tau^2\partial_t + \tau^3\partial_u$, be a vector field and $\Delta_1 = u_t + 2u^2u_x - u_x^2 - \frac{1}{2}u_{xx}u$ and $\Delta_2 = \tau^3 - \tau^1 u_x - \tau^2 u_t$. Without loss of generality we choose $\tau^2 = 1$, from Δ_2: $u_t = \tau^3 - \tau^1 u_x$. Now via substitute this equation in Δ_1 and $\tau^1 = A_1$, $\tau^3 = A_3$:

$$A_3 + (2u^2 - A_1)u_x - u_x^2 - \frac{1}{2}uu_{xx} = 0, \tag{13}$$

suppose $\mathbf{w} = L\partial_x + K\partial_t + M\partial_u$. applying $Pr^{(2)}\mathbf{w}$ to equation (13) determining equations yields:

$$2(A_1 - 2u^2)M_x + uM_{xx} = 0, \qquad 4uM - uM_{xu} + \frac{1}{2}uL_{xx} - 2M_x + (A_1 - 2u^2)(L_x - M_u) = 0,$$
$$2uL_x - uM_u - M = 0, \quad K_x = 0, \qquad 4(M_u - L_x) + uM_{uu} = 0, \qquad L_u = 0, \quad K_u = 0,$$

Now via solve upper equation and substitute $L = \tau^1$, $K = 1$, $M = \tau^3$, in $\mathbf{v}$, leads $\mathbf{v} = \partial_x + \partial_t$.

6. Symmetry reductions and invariant solution of FDE equation

In this section, according to the vector fields $\mathbf{v} = \partial_x - \partial_t$, and via change variable $z = x + t$ and $u = f(z)$ we find that: $u_t = du/dt = df(z)/dt = (df/dz)(dz/dt) = f_z$, $u_x = f_z$, and $u_{xx} = f_{zz}$; Substitude it in equation (13), leads $2f_z^2 - 2(1 + 2f^2)f_z + ff_{zz} = 0$.

7. Conservation Law of FDE equation

Consider a system $\mathbf{R}\{x; u\}$ of N differential equations of order k with n independent variables $x = (x^1, \cdots, x^n)$ and m dependent variables $u(x) = (u^1(x), \cdots, u^m(x))$, given by

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \cdots, \partial^k u) = 0, \quad \sigma = 1, \cdots, N. \tag{14}$$

Definition A *local conservation law* of the DE system (14) is a divergence expression

$$\mathrm{D}_1\Phi^1[u] + \cdots + \mathrm{D}_N\Phi^N[u] = 0. \tag{15}$$

holding for all solutions of the DE system (14). In (15), D_i and $\Phi^i[u] = \Phi^i(x, u, \partial u, \cdots, \partial^r u)$, $i = 1, \cdots, n$ respectively are total derivative operators and the *fluxes* of the conservation law. $i = 1, \cdots, n$, respectively are total derivative operators and the fluxes of the conservation law. A systematic way for the determination of conservation laws associated with variational symmetries for systems of Euler-Lagrange equations is in deed the famous Noether theorem.

7.1. Direct method to find multipliers of conservation law

In general, for a given non-degenerate DE system (14), nontrivial local conservation laws arise from seeking scalar products that involve linear combinations of the equations of the DE system (14) with *multipliers* (factors) that yield nontrivial divergence expressions. In seeking such expressions, the dependent variables and each of their derivatives that appear in the DE system (14) or in the multipliers, are replaced by arbitrary functions. Such divergence expressions vanish on all solutions of the DE system (14) provided the multipliers are non-singular. In particular a set of multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N = \{\Lambda_\sigma(x, U, \partial U, \cdots, \partial^l U)\}_{\sigma=1}^N$ yields a divergence expression for the DE system $R\{x; u\}$ (14) if the identity $\Lambda_\sigma[U]R^\sigma[U] \equiv \mathrm{D}_i\Phi^i[U]$, holds for *arbitrary* functions $U(x)$. Then on the solutions $U(x) = u(x)$ of the DE system (2), if $\Lambda_\sigma[u]$ is non-singular, one has a local conservation law $\Lambda_\sigma[u]R^\sigma[u] \equiv \sum \mathrm{D}_i\Phi^i[u] = 0$.

Definition. The *Euler operator* with respect to U^μ is the operator defined by $E_{U^\mu} = \partial/\partial U^\mu - \mathrm{D}_i\, \partial/\partial U_i^\mu + \cdots + (-1)^s \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_s}\, \partial/\partial U^\mu_{i_1\cdots i_s} + \cdots$. We know that the *Euler operators* annihilate any divergence expression. Now we use this subject to find local conservation law multiplier. We seek all local conservation law multipliers of the form $\Lambda(x, t, u, u_x)$ of the FDE equation. Now suppose $\Lambda = \Lambda(x, t, U, U_x)$,$\Lambda$ is a local conservation law multiplier of the FDE equation if and only if

$$E_U\left(\Lambda(U_t + 2U^2U_x - U_x^2 - \frac{1}{2}UU_{xx})\right) \equiv 0. \tag{16}$$

Then from equations (15) and (16), and splits into the equations:

$$\Lambda_{U_x} = 0, \quad \Lambda - U\Lambda_U = 0, \quad UU_x^2\Lambda_{UU} + 2\Lambda_t - 2U_x\Lambda_x + \Lambda_{xx}U + 4U^2\Lambda_x + 2UU_x\Lambda_{xU} = 0, \tag{17}$$

whose solution yields the one local conservation law multiplier $\Lambda = U$.

7.2. Computation fluxes of conservation laws

In this part we find fluxes of conservation laws and use the direct method of flux computation. For the multiplier $\Lambda = U$,

$$U(u_t + 2u^2u_x - u_x^2 - \frac{1}{2}uu_{xx}) = \mathrm{D}_x\Psi + \mathrm{D}_t\Phi, \tag{18}$$

(D_t denote total derivative operators), assume $\Psi = \Psi(x, t, U, U_x)$,$\Phi = \Phi(x, t, U, U_x)$ now we find Φ, Ψ. via expand the equation (18) we obtain $\Psi_x + \Psi_U U_x + \Psi_{U_x}U_{xx} + \Phi_t + \Phi_U U_t + \Phi_{U_x}U_{xt} = U(U_t + 2U^2U_x - {U_x}^2 - UU_{xx}/2)$. Matching the terms of the highest order derivatives U_t,U_{xx} finds that

$$\Psi_x + \Psi_U U_x + \Phi_t + UU_x^2 + \Phi_{U_x}U_{xt} - 2U_xU^3 = 0 \qquad 2\Psi_{U_x} + U^2 = 0, \qquad \Phi_U = U, \tag{19}$$

via solving this equation we obtain: $\Phi = U^2/2 + F_1(x, t, U_x)$ and $\Psi = -U^2U_x/2 + F_2(x, t, U)$, where $F_1(x, t, U_x)$ and $F_2(x, t, U)$ are arbitrary function, via Substituting (18) into the determining equations obtain $\Phi = U^2/2$, and $\Psi = -U^2U_x/2 + U^4/2$. Finally

$$U(U_t + 2U^2U_x - U_x^2 - \frac{1}{2}UU_{xx}) = \mathrm{D}_t(\frac{1}{2}U^2) + \mathrm{D}_x(-\frac{1}{2}U^2U_x + \frac{1}{2}U^4). \tag{20}$$

7.3. **Computation flux of Conservation Laws via the First Homotopy Method**

Definition. (higher Euler operators) The continuous higher Euler operators for one-dimensional (1D) are defined by $\mathcal{L}^{(i)}_{\mathbf{u}(x)} = \sum_{i=1}^\infty \binom{k}{i}(-\mathrm{D}_x)^{k-i}\, \partial/\partial u_{kx}$. Also for two-dimensional (2D) are defined by

$$\mathcal{L}^{(i_x,i_t)}_{\mathbf{u}(x,t)} = \sum_{k_x=i_x}^\infty \sum_{k_t=i_t}^\infty \binom{k_x}{i_x}\binom{k_t}{i_t}(-\mathrm{D}_x)^{k_x-i_x}(-\mathrm{D}_t)^{k_t-i_t}\frac{\partial}{\partial u_{k_x x k_t t}}, \tag{21}$$

for example we expand above equation

$$\begin{aligned}
\mathcal{L}^{(1,0)}_{u(x,t)} &= \frac{\partial}{\partial u_x} - 2\mathrm{D}_x\frac{\partial}{\partial u_{2x}} - \mathrm{D}_t\frac{\partial}{\partial u_{xt}} + 3\mathrm{D}_x^2\frac{\partial}{\partial u_{3x}} + \cdots,\\
\mathcal{L}^{(0,1)}_{u(x,t)} &= \frac{\partial}{\partial u_t} - 2\mathrm{D}_t\frac{\partial}{\partial u_{2t}} - \mathrm{D}_x\frac{\partial}{\partial u_{tx}} + 3\mathrm{D}_t^2\frac{\partial}{\partial u_{3t}} + \cdots,\\
\mathcal{L}^{(1,1)}_{u(x,t)} &= \frac{\partial}{\partial u_{xt}} - 2\mathrm{D}_x\frac{\partial}{\partial u_{2xt}} - 2\mathrm{D}_t\frac{\partial}{\partial u_{x2t}} + 3\mathrm{D}_x^2\frac{\partial}{\partial u_{3xt}}.
\end{aligned}$$

Definition. (homotopy operator) The continuous homotopy operator is defined by

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,t)}(f) = \int_0^1 \sum_{j=1}^{N} I^{(x)}_{u_j}(f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}, \qquad \mathcal{H}^{(t)}_{\mathbf{u}(x,t)}(f) = \int_0^1 \sum_{j=1}^{N} I^{(t)}_{u_j}(f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda},$$

with

$$I^{(x)}_{u_j}(f) = \sum_{i_x=0}^{\infty} \sum_{i_t=0}^{\infty} \binom{1+i_x+i_t}{1+i_x} \mathrm{D}_x^{i_x} \mathrm{D}_t^{i_t} \left(u_j L^{(1+i_x, i_t)}_{u_j(x,t)}(f) \right)$$

Theorem. Suppose $R[U]$ is a divergence expression $R[U] = \boldsymbol{Div}\,\Phi^i[U] = \sum \mathrm{D}_i \Phi^i[U] = 0$ and $R[0] = 0$. Then the fluxes $\Phi^i[U]$ are given by $\Phi^i[u] = \mathcal{H}(R[U])$. □

Now for FDE equation we obtain :

$$\mathcal{L}^{(1,0)}_{u(x,t)} = 2u^3, \quad \mathcal{L}^{(0,1)}_{u(x,t)} = u, \quad \mathcal{L}^{(2,0)}_{u(x,t)} = -\frac{1}{2}u^2, \quad \mathcal{L}^{(1,1)}_{u(x,t)} = 0, \quad \mathcal{L}^{(0,2)}_{u(x,t)} = 0, \tag{22}$$

now

$$H^{(x)}_{\mathbf{u}(x,t)}(f) = \int_0^1 (2\lambda^3 u^4 - \frac{3}{2}\lambda^2 u^2 u_x) d\lambda = \frac{1}{2}u^2(u^2 - u_x), \qquad H^{(t)}_{\mathbf{u}(x,t)}(f) = \int_0^1 \lambda u^2 d\lambda = \frac{1}{2}u^2.$$

So, the conservation law corresponding to the local multiplier$\Lambda = U$ for the FDE equation is concluded as $\mathrm{D}_t(u^2) + \mathrm{D}_x(u^4 - u^2 u_x) = 0$.

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Please cite this article using:

Mehdi Nadjafikhah, Omid Chekini, Conservation law and Lie symmetry analysis of Foam Drainage equation, AUT J. Math. Com., 2(1) (2021) 37-44
DOI: 10.22060/ajmc.2020.18460.1036