AUT Journal of Mathematics and Computing

AUT J. Math. Com., 2(1) (2021) 17-25
DOI: 10.22060/ajmc.2020.18833.1041

Original Article

A class of operator related weighted composition operators between Zygmund space

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ABSTRACT: Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. Let $u, v \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. A class of operator related weighted composition operators is defined as follow

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

In this work, we obtain some new characterizations for boundedness and essential norm of operator $T_{u,v,\varphi}$ between Zygmund space.

Review History:

Received:06 August 2020
Accepted:25 September 2020
Available Online:01 February 2021

Keywords:

Boundedness
Compactness
Essential norm
Zygmund space

AMS Subject Classification (2010):

16W25; 46L57; 47B47

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. For a function $u \in H(\mathbb{D})$ and analytic self-map φ of $\mathbb{D}(\varphi(\mathbb{D}) \subset \mathbb{D})$, the weighted composition operator uC_φ on $H(\mathbb{D})$ is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

When $u \equiv 1$, we get the composition operator C_φ, which is defined by $C_\varphi(f) = f \circ \varphi$. For more information about weighted composition operators see [1, 2, 13, 14].

Let $u, v \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. S. Stević and co-authors in [11] defined the operator $T_{u,v,\varphi}$ as follows

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let D denote the differentiation operator then $T_{u,v,\varphi} = uC_\varphi + vC_\varphi D$. More information about the operator $T_{u,v,\varphi}$ can be found in [4, 7, 11, 10, 15, 16, 17]. Product-type operators on some spaces of analytic functions on the unit disk and the unit ball or the upper half-plane have become a subject of increasing interest in the last five years (see, e.g., the following representative papers [3, 5, 6, 9], and the related references therein).

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A function $f \in H(\mathbb{D})$ is said to be in the Zygmund space $\mathcal{Z}$, if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty.$$

The space $\mathcal{Z}$ becomes a Banach space with the following norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty.$$

The little Zygmund space $\mathcal{Z}_0$, is a closed subspace of $\mathcal{Z}$, consists of all function $f \in \mathcal{Z}$ for which $\lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0$.

From [18, Proposition 8], we get the next lemma.

Lemma 1.1. *For any $f \in \mathcal{Z}$ and $n \in \mathbb{N}$,*

$$\|f\|_{\mathcal{Z}} \approx \sum_{i=0}^{n} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{n+1}|f^{(n+2)}(z)|.$$

Lemma 1.2 ([14]). *Let $f \in \mathcal{Z}$. Then,*

$$|f(z)| \le \|f\|_{\mathcal{Z}} \quad \text{and} \quad |f'(z)| \preceq \|f\|_{\mathcal{Z}} \log\frac{2}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

Lemma 1.3 ([14]). *Let $\{f_n\}$ be a bounded sequence in $\mathcal{Z}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Then*

$$\limsup_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n(z)| = 0.$$

Recall that the essential norm of a continuous linear operator $T : X \to Y$ is the distance from T to the compact operators, that is

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\| : K : X \to Y \text{ is compact}\}.$$

Here X and Y are Banach spaces. Notice that $\|T\|_e = 0$ if and only if T is compact.

Recently, Liu and Yu in [7] studied the boundedness and compactness of operator $T_{u,v,\varphi}$ from the Besov spaces into the weighted-type space H^∞_μ. In this work, we find some characterizations for boundedness and essential norm of operator $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$. As some applications, we get some new characterizations of the boundedness, essential norm and compactness of weighted composition operators between Zygmund space.

Throughout this paper, we say that $A \succeq B$, if there exists a constant C such that $A \ge CB$. The symbol $A \approx B$ means that $A \succeq B \succeq A$.

2. Boundedness

In this section, the boundedness of operator $T_{u,v,\varphi}$ between Zygmund spaces is characterized. We begin with the next lemma.

Lemma 2.1. *Let φ be an analytic self-map of $\mathbb{D}$. Then for any $a \in \mathbb{D}$, there exists a function Ψ_a in $\mathcal{Z}_0$ such that $\sup_{a\in\mathbb{D}}\|\Psi_a\|_{\mathcal{Z}} < \infty$ and*

$$\Psi_a(\varphi(a)) = \Psi_a''(\varphi(a)) = \Psi_a'''(\varphi(a)) = 0 \quad \text{and} \quad \Psi_a'(\varphi(a)) = \log\frac{2}{1-|\varphi(a)|^2}.$$

Proof. If $\varphi(a) = 0$, then set $\Psi_a(z) = \int_0^z \log 2\, d\xi$, as desired. For any $a \in \mathbb{D}$ with $\varphi(a) \ne 0$ and $k \in \{1, 2, 3\}$, set

$$h_{a,k}(z) = \frac{(k+3)!}{k!} + \int_{\varphi(a)}^{z}\Big((3+k)\log\frac{2}{1-\overline{\varphi(a)}\xi} - \frac{(\log\frac{2}{1-\overline{\varphi(a)}\xi})^{3+k}}{(\log\frac{2}{1-|\varphi(a)|^2})^{2+k}}\Big)d\xi.$$

It is obvious that $h_{a,k} \in \mathcal{Z}_0$. In this case

$$\Psi_a(z) = 5h_{a,1}(z) - 6h_{a,2}(z) + 2h_{a,3}(z)$$

as desired. By simple calculation, we get

$$\sup_{a\in\mathbb{D}}\|h_{a,k}\|_{\mathcal{Z}} < \infty \quad k \in \{1, 2, 3\}.$$

Hence, $\sup_{a\in\mathbb{D}}\|\Psi_a\|_{\mathcal{Z}} < \infty$. $\square$

For simplicity in calculation, we set

$$\widetilde{A_0}(z) = (1-|z|^2)|u''(z)|, \qquad \widetilde{A_1}(z) = (1-|z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)|$$
$$\widetilde{A_2}(z) = (1-|z|^2)|u(z)\varphi'^2(z) + 2v'(z)\varphi'(z) + v(z)\varphi''(z)|, \qquad \widetilde{A_3}(z) = (1-|z|^2)|v(z)\varphi'^2(z)|. \tag{1}$$

Theorem 2.2. *Let $u \in \mathcal{Z}$, $v \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) The operator $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ is bounded.

(b) The operator $T_{u,v,\varphi} : \mathcal{Z}_0 \to \mathcal{Z}$ is bounded.

(c) $\max\left\{\sup_{z\in\mathbb{D}} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2},\ \sup_{j\geq 1} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\right\} < \infty.$

(d) $\max\left\{\sup_{z\in\mathbb{D}} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2},\quad \sup_{a\in\mathbb{D}} \|T_{u,v,\varphi} f_{1,a}\|_{\mathcal{Z}}\right\} < \infty$ *and*

$$\max\left\{\sup_{a\in\mathbb{D}} \|T_{u,v,\varphi} f_{2,a}\|_{\mathcal{Z}} < \infty, \quad \sup_{z\in\mathbb{D}} \widetilde{A_2}(z), \quad \sup_{z\in\mathbb{D}} \widetilde{A_3}(z)\right\} < \infty,$$

where $f_{i,a}(z) = \frac{(1-|a|^2)^{i+2}}{(1-\overline{a}z)^{i+1}}$.

(e)

$$\max\left\{\sup_{z\in\mathbb{D}} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2},\ \sup_{z\in\mathbb{D}} \frac{\widetilde{A_2}(z)}{(1-|\varphi(z)|^2)},\ \sup_{z\in\mathbb{D}} \frac{\widetilde{A_3}(z)}{(1-|\varphi(z)|^2)^2}\right\} < \infty.$$

Proof. $(a) \Rightarrow (b)$ It is obvious.

$(b) \Rightarrow (c)$ For any $a \in \mathbb{D}$, let Ψ_a be the function defined in Lemma 2.1.

$$\widetilde{A_1}(a) \log \frac{2}{1-|\varphi(a)|^2} = \widetilde{A_1}(a)|\Psi_a'(\varphi(a))| = (1-|a|^2)|(T_{u,v,\varphi}\Psi_a)''(a)| \leq \|T_{u,v,\varphi}\Psi_a\|_{\mathcal{Z}}$$
$$\leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} \sup_{a\in\mathbb{D}} \|\Psi_a\|_{\mathcal{Z}} < \infty.$$

From [8], we know that the sequence $\{z^j\}_0^\infty$ is bounded in $\mathcal{B}_0$, hence $\{j^{-1}z^j\}_1^\infty$ is bounded in $\mathcal{Z}_0$, therefore

$$\sup_{j\geq 1} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} \sup_{j\geq 1} \|z^j\|_{\mathcal{B}} < \infty.$$

$(c) \Rightarrow (d)$ Let $p_j(z) = z^j$. For any $a \in \mathbb{D}$ and $i = 1, 2$, we have

$$\|T_{u,v,\varphi} f_{i,a}\|_{\mathcal{Z}} \leq (1-|a|^2)^{i+1}\left(\|u\|_{\mathcal{Z}} + \sum_{j=1}^{\infty} \binom{i+j-1}{j} j|a|^j j^{-1}\|T_{u,v,\varphi} p_j\|_{\mathcal{W}_\mu^n}\right)$$
$$\leq (1+i2^{i+1}) \max\{\|u\|_{\mathcal{Z}},\ \sup_{j\geq 1} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\}.$$

Since a is arbitrary, $\sup_{a\in\mathbb{D}} \| T_{u,v,\varphi} f_{i,a} \|_{\mathcal{Z}} < \infty$. Applying the operator $T_{u,v,\varphi}$ for $p_2(z) = z^2$, so

$$\widetilde{A_2}(z) \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} + \widetilde{A_0}(z)|\varphi(z)|^2 + 2\widetilde{A_1}(z)|\varphi(z)| \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} + \sup_{z\in\mathbb{D}} \widetilde{A_0}(z) + 2\sup_{z\in\mathbb{D}} \widetilde{A_1}(z) < \infty.$$

Therefore, $\sup_{z\in\mathbb{D}} \widetilde{A_2}(z) < \infty$. Similarly by applying the operator $T_{u,v,\varphi}$ for $p_3(z) = z^3$, we get

$$\widetilde{A_3}(z) \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} + \sup_{z\in\mathbb{D}} \widetilde{A_0}(z) + 3\sup_{z\in\mathbb{D}} \widetilde{A_1}(z) + 6\sup_{z\in\mathbb{D}} \widetilde{A_2}(z) < \infty.$$

Hence $\sup_{z\in\mathbb{D}} \widetilde{A_3}(z) < \infty$.

$(d) \Rightarrow (e)$ Suppose that (d) holds. we set

$$k_{1,a}(z) = \frac{5}{6} f_{1,a}(z) - \frac{1}{3} f_{2,a}(z). \tag{2}$$

Let $|\varphi(a)| > \frac{1}{2}$, so

$$\frac{\widetilde{A_2}(a)|\varphi(a)|^2}{(1-|\varphi(a)|^2)} \leq \sup_{a\in\mathbb{D}} \|T_{u,v,\varphi}k_{1,\varphi(a)}\|_{\mathcal{Z}} + \sup_{a\in\mathbb{D}} \widetilde{A_0}(a)(1-|\varphi(a)|^2) + \sup_{a\in\mathbb{D}} \widetilde{A_1}(a)|\varphi(a)|$$
$$\preceq \sup_{a\in\mathbb{D}} \|f_{1,a}\|_{\mathcal{Z}} + \sup_{a\in\mathbb{D}} \|f_{2,a}\|_{\mathcal{Z}} + \sup_{a\in\mathbb{D}} \widetilde{A_0}(a) + \sup_{a\in\mathbb{D}} \widetilde{A_1}(a) < \infty.$$

From previous inequality, $\sup_{|\varphi(a)|>\frac{1}{2}} \frac{\widetilde{A_2}(a)}{(1-|\varphi(a)|^2)} < \infty$. Also by using (d), we obtain

$$\sup_{|\varphi(a)|\leq\frac{1}{2}} \frac{\widetilde{A_2}(a)}{(1-|\varphi(a)|^2)} \leq \frac{4}{3} \sup_{|\varphi(a)|\leq\frac{1}{2}} \widetilde{A_2}(a) < \infty.$$

Now we set

$$k_{2,a}(z) = \frac{-1}{6} f_{1,a}(z) + \frac{1}{12} f_{2,a}(z). \tag{3}$$

Let $|\varphi(a)| > \frac{1}{2}$, hence

$$\frac{\widetilde{A_3}(a)|\varphi(a)|^3}{(1-|\varphi(a)|^2)^2} \leq \sup_{a\in\mathbb{D}} \|T_{u,v,\varphi}k_{2,\varphi(a)}\|_{\mathcal{Z}} + \sup_{a\in\mathbb{D}} \widetilde{A_0}(a)(1-|\varphi(a)|^2) + \sup_{a\in\mathbb{D}} \widetilde{A_1}(a)|\varphi(a)|$$
$$\preceq \sup_{a\in\mathbb{D}} \|f_{1,a}\|_{\mathcal{Z}} + \sup_{a\in\mathbb{D}} \|f_{2,a}\|_{\mathcal{Z}} + \sup_{a\in\mathbb{D}} \widetilde{A_0}(a) + \sup_{a\in\mathbb{D}} \widetilde{A_1}(a) < \infty.$$

So, $\sup_{|\varphi(a)|>\frac{1}{2}} \frac{\widetilde{A_3}(a)}{(1-|\varphi(a)|^2)^2} < \infty$. Also from (d), we have

$$\sup_{|\varphi(a)|\leq\frac{1}{2}} \frac{\widetilde{A_3}(a)}{(1-|\varphi(a)|^2)^2} \leq \frac{16}{9} \sup_{|\varphi(a)|\leq\frac{1}{2}} \widetilde{A_3}(a) < \infty.$$

$(e) \Rightarrow (a)$ Let f be arbitrary function in $\mathcal{Z}$. Using Lemmas 1.1 and 1.2, we have

$$(1-|z|^2|(T_{u,v,\varphi}f)''(z)| \preceq \|f\|_{\mathcal{Z}}\|u\|_{\mathcal{Z}} + \|f\|_{\mathcal{Z}} \sup_{z\in\mathbb{D}} \widetilde{A_1}(z) \log\frac{2}{1-|\varphi(z)|^2}$$
$$+\|f\|_{\mathcal{Z}} \sup_{z\in\mathbb{D}} \frac{\widetilde{A_2}(z)}{(1-|\varphi(z)|^2)} + \|f\|_{\mathcal{Z}} \sup_{z\in\mathbb{D}} \frac{\widetilde{A_3}(z)}{(1-|\varphi(z)|^2)^2}.$$

Also

$$|(T_{u,v,\varphi}f)(0)| \leq \|f\|_{\mathcal{Z}}\Big(|u(0)| + |v(0)| \log\frac{2}{1-|\varphi(0)|^2}\Big)$$

and

$$|(T_{u,v,\varphi}f)'(0)| \leq \|f\|_{\mathcal{Z}}\Big(|u'(0)| + |u(0)\varphi'(0) + v'(0)| \log\frac{2}{1-|\varphi(0)|^2} + \frac{|v(0)|}{1-|\varphi(0)|^2}\Big)$$

Thus, $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ is bounded. The proof is completed. □

3. Essential norm

In this section, some estimates for the essential norm of operator $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ are obtained. For the study of the essential norm, we need the following lemma, which can be proved in a standard way, see, for example [12, Lemma 2.10].

Lemma 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ and $S : \mathcal{Z}(\mathcal{Z}_0) \to \mathcal{Z}$ be bounded. Then S is compact if and only if whenever $\{f_k\}$ is bounded in $\mathcal{Z}(\mathcal{Z}_0)$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{k\to\infty} \|Sf_k\|_{\mathcal{Z}} = 0.$$

Lemma 3.2. *Let $u, v \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, such taht $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ be bounded. Then*

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0\to\mathcal{Z}} \geq \limsup_{|\varphi(z)|\to 1} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2}.$$

where $\widetilde{A_1}(z)$ is defined in (1).

Proof. Let $\{z_i\}$ be a sequence in $\mathbb{D}$ such that $\lim |\varphi(z_i)| = 1$. We assume that for each i, $\varphi(z_i) \neq 0$. First we show that there exists a bounded sequence $\{\Psi_i\}$ in $\mathcal{Z}_0$ such that, $\{\Psi_i\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$ and

$$\Psi_i(\varphi(z_i)) = \Psi_i''(\varphi(z_i)) = \Psi_i^{'''}(\varphi(z_i)) = 0, \; \Psi_i'(\varphi(z_i)) = \log \frac{2}{1-|\varphi(z_i)|^2}.$$

For each i and $k \in \{1, 2, 3\}$, we set

$$h_{i,k}(z) = (k+1)(k+2)(k+3) + \int_{\varphi(z_i)}^{z} \Big((3+k)\frac{\big(\log \frac{2}{1-\overline{\varphi(z_i)}\xi}\big)^{2+k}}{\big(\log \frac{2}{1-|\varphi(z_i)|^2}\big)^{1+k}} - (2+k)\frac{\big(\log \frac{2}{1-\overline{\varphi(z_i)}\xi}\big)^{3+k}}{\big(\log \frac{2}{1-|\varphi(z_i)|^2}\big)^{2+k}}\Big)d\xi.$$

It is clear that $h_{i,k} \in \mathcal{Z}_0$. Now the following sequence give us all mentioned properties

$$\Psi_i(z) = \frac{1}{12}h_{i,1}(z) - \frac{3}{20}h_{i,2}(z) + \frac{1}{15}h_{i,3}(z).$$

Let $K : \mathcal{Z}_0 \to \mathcal{Z}$ be arbitrary compact operator. By using Lemma 3.1, we have

$$\begin{aligned}\|(T_{u,v,\varphi} - K)\Psi_i\|_{\mathcal{Z}\to\mathcal{Z}} &\geq \|(T_{u,v,\varphi} - K)\Psi_i\|_{\mathcal{Z}_0\to\mathcal{Z}} \geq \limsup_{i\to\infty} \|T_{u,v,\varphi}\Psi_i\|_{\mathcal{Z}} - \limsup_{i\to\infty} \|K\Psi_i\|_{\mathcal{Z}} \\ &\geq \limsup_{i\to\infty} \widetilde{A_1}(z_i) \log \frac{2}{1-|\varphi(z_i)|^2} = \limsup_{|\varphi(z)|\to 1} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2}.\end{aligned}$$

Based on the defination of essential norm, we have

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0\to\mathcal{Z}} = \inf_K \|T_{u,v,\varphi} - K\| \geq \limsup_{|\varphi(z)|\to 1} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2}.$$

The proof is completed. □

Theorem 3.3. *Let $u, v \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ and $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ be bounded. Then*

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}} \approx \max\{\rho_i\}_{i=0}^2 \approx \max\{\sigma_i\}_{i=0}^2$$

where

$$f_{i,a}(z) = \frac{(1-|a|^2)^{i+2}}{(1-\overline{a}z)^{i+1}}, \qquad \sigma_0 = \rho_0 = \limsup_{|\varphi(z)|\to 1} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2}, \quad \sigma_1 = \limsup_{|a|\to 1} \|T_{u,v,\varphi} f_{1,a}\|_{\mathcal{Z}},$$

$$\sigma_2 = \limsup_{|a|\to 1} \|T_{u,v,\varphi} f_{2,a}\|_{\mathcal{Z}}, \quad \rho_1 = \limsup_{|\varphi(z)|\to 1} \frac{\widetilde{A_2}(z)}{(1-|\varphi(z)|^2)}, \qquad \rho_2 = \limsup_{|\varphi(z)|\to 1} \frac{\widetilde{A_3}(z)}{(1-|\varphi(z)|^2)^2}$$

and $\widetilde{A_0}(z), \widetilde{A_1}(z), \widetilde{A_2}(z), \widetilde{A_3}(z)$ are defined in (1).

Proof. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim |\varphi(z_j)| = 1$ and $k_{i,a} (i = 1, 2)$ be functions are defined in (2) and (3). It is clear that for all $a \in \mathbb{D}$, $\|k_{i,a}\|_{\mathcal{Z}} \preceq 1 (i = 1, 2)$ and if $a \neq 0$ then $k_{i,a} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. So, by using Lemma 3.1 for any compact operator K from $\mathcal{Z}$ into $\mathcal{Z}$, we get

$$\begin{aligned}\|T_{u,v,\varphi} - K\|_{\mathcal{Z}\to\mathcal{Z}} &\succeq \limsup_{j\to\infty} \|(T_{u,v,\varphi} - K)k_{i,\varphi(z_j)}\|_{\mathcal{Z}} \geq \limsup_{j\to\infty} \|T_{u,v,\varphi}k_{i,\varphi(z_j)}\|_{\mathcal{Z}} - \limsup_{j\to\infty} \|Kk_{i,\varphi(z_j)}\|_{\mathcal{Z}} \\ &\geq \limsup_{j\to\infty} \frac{\widetilde{A_{i+1}}(z_j)|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^i} - \limsup_{j\to\infty} \widetilde{A_0}(z_j)(1-|\varphi(z_j)|^2) - \limsup_{j\to\infty} \widetilde{A_1}(z_j)|\varphi(z_j)|.\end{aligned}$$

From Theorem 2.2(c), we have $\limsup_{|\varphi(z)|\to 1} \widetilde{A_1}(z) = 0$, so from previous inequality and definaition of essential norm, we have

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}} = \inf_K \|T_{u,v,\varphi} - K\|_{\mathcal{Z}\to\mathcal{Z}} \succeq \max\{\rho_1, \rho_2\}.$$

Using Lemma 3.2, we get

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}} \succeq \max\{\rho_0, \rho_1, \rho_2\}.$$

Also $\|f_{i,a}\|_{\mathcal{Z}} \preceq 1 (i = 1, 2)$ and $f_{i,a} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. So, for any compact operator $K : \mathcal{Z} \to \mathcal{Z}$, by Lemma 3.1, we get $\limsup_{|a|\to 1} \|Kf_{i,a}\|_{\mathcal{Z}} = 0$. Hence

$$\|T_{u,v,\varphi} - K\|_{\mathcal{Z}\to\mathcal{Z}} \succeq \limsup_{|a|\to 1} \|(T_{u,v,\varphi} - K)f_{i,a}\|_{\mathcal{Z}} \geq \limsup_{|a|\to 1} \|T_{u,v,\varphi}f_{i,a}\|_{\mathcal{Z}} - \limsup_{|a|\to 1} \|Kf_{i,a}\|_{\mathcal{Z}} = \sigma_i.$$

By the last inequality and Lemma 3.2, we obtain

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}} = \inf_K \|T_{u,v,\varphi} - K\|_{\mathcal{Z}\to\mathcal{Z}} \succeq \max\{\sigma_0, \sigma_1, \sigma_2\}.$$

Now, we prove that

$$\min\{\max\{\sigma_i\}_{i=0}^2, \max\{\rho_i\}_{i=0}^2\} \succeq \|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}}.$$

For $r \in [0, 1)$, we define $K_r f(z) = f_r(z) = f(rz)$. It is clear that $K_r : \mathcal{Z} \to \mathcal{Z}$ is a compact operator with $\|K_r\| \leq 1$. Also we khow that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for any positive integer j, the operator $T_{u,v,\varphi}K_{r_j} : \mathcal{Z} \to \mathcal{Z}$ is compact. So

$$\limsup_{j\to\infty} \|T_{u,v,\varphi} - T_{u,v,\varphi}K_{r_j}\| \geq \|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}}.$$

Therefore, based on the defination of essential norm it is enough to prove that

$$\min\{\max\{\sigma_i\}_{i=0}^2, \max\{\rho_i\}_{i=0}^2\} \succeq \limsup_{j\to\infty} \|T_{u,v,\varphi} - T_{u,v,\varphi}K_{r_j}\|_{\mathcal{Z}\to\mathcal{Z}}.$$

Let f be arbitrary function in $\mathcal{Z}$ such that $\|f\|_{\mathcal{Z}} \leq 1$,

$$\begin{aligned}
\|(T_{u,v,\varphi} - T_{u,v,\varphi}K_{r_j})f\|_{\mathcal{Z}} \leq\ & \underbrace{(|u(0)| + |u'(0)|)|(f - f_{r_j})(\varphi(0))|}_{R_1} + \underbrace{|v(0)\varphi'(0)||(f - f_{r_j})''(\varphi(0))|}_{R_2} \\
& + \underbrace{(|v(0)| + |u(0)\varphi'(0)| + |v'(0)|)(f - f_{r_j})'(\varphi(0))|}_{R_3} + \underbrace{\sup_{z\in\mathbb{D}} \widetilde{A_0}(z)|(f - f_{r_j})(\varphi(z))|}_{L_0} + \underbrace{\sup_{|\varphi(z)|\leq r_N} \widetilde{A_1}(z)|(f - f_{r_j})'(\varphi(z))|}_{L_{11}} \\
& + \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_1}(z)|(f - f_{r_j})'(\varphi(z))|}_{L_{12}} + \underbrace{\sup_{|\varphi(z)|\leq r_N} \widetilde{A_2}(z)|(f - f_{r_j})''(\varphi(z))|}_{L_{21}} + \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_2}(z)|(f - f_{r_j})''(\varphi(z))|}_{L_{22}} \\
& + \underbrace{\sup_{|\varphi(z)|\leq r_N} \widetilde{A_3}(z)|(f - f_{r_j})'''(\varphi(z))|}_{L_{31}} + \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_3}(z)|(f - f_{r_j})'''(\varphi(z))|}_{L_{32}}
\end{aligned} \tag{4}$$

where $N \in \mathbb{N}$ and $r_j \geq \frac{1}{2}$ for all $j \geq N$. Since for any $k \in \mathbb{N}_0$, $(f - f_{r_j})^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, from Lemmas 1.2, 1.3 and Theorem 2.2 (d), we get

$$\limsup_{j\to\infty} R_t = \limsup_{j\to\infty} L_0 = \limsup_{j\to\infty} L_{t1} = 0 \qquad (t = 1, 2, 3). \tag{5}$$

Also

$$L_{12} \leq \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_1}(z)|f^{'}(\varphi(z))|}_{L_{12}^1} + \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_1}(z)|r_j f^{'}(r_j\varphi(z))|}_{L_{12}^2}. \tag{6}$$

No we obtain estimate for L_{12}^1. Using Lemma 1.2,

$$L_{12}^1 = \sup_{|\varphi(z)|>r_N} \widetilde{A_1}(z)|f'(\varphi(z))| \preceq \sup_{|\varphi(z)|>r_N} \widetilde{A_1}(z)\|f\|_{\mathcal{Z}} \log \frac{2}{1-|\varphi(z)|^2}.$$

Letting $N \to \infty$, we get

$$\limsup_{j\to\infty} L_{12}^1 \preceq \sigma_0 = \rho_0. \tag{7}$$

Similarly, we have

$$\limsup_{j\to\infty} L_{12}^2 \preceq \sigma_0 = \rho_0. \tag{8}$$

On the other hand

$$\begin{aligned} L_{22} &\leq \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_2}(z)|f''(\varphi(z))|}_{L_{22}^1} + \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_2}(z)|r_j f''(r_j\varphi(z))|}_{L_{22}^2}, \\ L_{32} &\leq \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_3}(z)|f'''(\varphi(z))|}_{L_{32}^1} + \underbrace{\sup_{|\varphi(z)|>r_N} \widetilde{A_3}(z)|r_j f'''(r_j\varphi(z))|}_{L_{32}^2}. \end{aligned} \tag{9}$$

Now we estimate $L_{2s}^1 (s = 2, 3)$. From Lemmas 1.1, 1.2, and (2) and (3),

$$\begin{aligned} L_{s2}^1 &= \sup_{|\varphi(z)|>r_N} \frac{(1-|\varphi(z)|^2)^{s-1}|f^{(s)}(\varphi(z))|}{|\varphi(z)|^s} \frac{|\varphi(z)|^s \widetilde{A_s}(z)}{(1-|\varphi(z)|^2)^{s-1}} \preceq \|f\|_{\mathcal{Z}} \sup_{|\varphi(z)|>r_N} \|T_{u,v,\varphi} k_{s,\varphi(z)}\|_{\mathcal{Z}} \\ &\preceq \sup_{|a|>r_N} \|T_{u,v,\varphi} f_{1,a}\|_{\mathcal{Z}} + \sup_{|a|>r_N} \|T_{u,v,\varphi} f_{2,a}\|_{\mathcal{Z}}. \end{aligned}$$

Letting $N \to \infty$, we get

$$\limsup_{j\to\infty} L_{s2}^1 \preceq \rho_{s-1} \quad \text{and} \quad \limsup_{j\to\infty} L_{s2}^1 \preceq \max\{\sigma_1, \sigma_2\}. \tag{10}$$

Similarly, we have

$$\limsup_{j\to\infty} L_{s2}^2 \preceq \rho_{s-1} \quad \text{and} \quad \limsup_{j\to\infty} L_{s2}^2 \preceq \max\{\sigma_1, \sigma_2\}. \tag{11}$$

By using (4), (5), (6), (7), (8), (9), (10) and (11), we obtain

$$\max\{\sigma_0, \sigma_1, \sigma_2\} \succeq \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{Z}}\leq 1} \|(T_{u,v,\varphi} - T_{u,v,\varphi}K_{r_j})f\|_{\mathcal{Z}} = \limsup_{j\to\infty} \|T_{u,v,\varphi} - T_{u,v,\varphi}K_{r_j}\|_{\mathcal{Z}\to\mathcal{Z}}$$

and

$$\max\{\rho_0, \rho_1, \rho_2\} \succeq \limsup_{j\to\infty} \|T_{u,v,\varphi} - T_{u,v,\varphi}K_{r_j}\|_{\mathcal{Z}\to\mathcal{Z}}.$$

Hence,

$$\min\{\max\{\sigma_i\}_{i=0}^2, \max\{\rho_i\}_{i=0}^2\} \succeq \limsup_{j\to\infty} \|T_{u,v,\varphi} - T_{u,v,\varphi}K_{r_j}\|_{\mathcal{Z}\to\mathcal{Z}}.$$

The proof is completed. □

Theorem 3.4. *Let $u, v \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ and $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ be bounded. Then*

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}} \approx \|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0\to\mathcal{Z}} \approx \max\{\limsup_{|\varphi(z)|\to 1} \widetilde{A_1}(z) \log \frac{2}{1-|\varphi(z)|^2},\ \limsup_{j\to\infty} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\}$$

where $\widetilde{A_1}(z)$ is defined in (1).

Proof. Let $p_j(z) = z^j$. It is clear that $\{j^{-1}p_j\}_1^\infty \subset \mathcal{Z}_0$ and $j^{-1}p_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. So, for any compact operator $K : \mathcal{Z}_0 \to \mathcal{Z}$, we have $\lim_{j\to\infty} j^{-1}\|Kp_j\|_{\mathcal{Z}} = 0$. Thus,

$$\begin{aligned}\|T_{u,v,\varphi} - K\|_{\mathcal{Z}_0\to\mathcal{Z}} &\succeq \limsup_{j\to\infty} j^{-1}\|(T_{u,v,\varphi} - K)p_j\|_{\mathcal{Z}} \limsup_{j\to\infty} j^{-1}\|T_{u,v,\varphi}p_j\|_{\mathcal{Z}} - \limsup_{j\to\infty} j^{-1}\|Kp_j\|_{\mathcal{Z}}\\ &= \limsup_{j\to\infty} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}.\end{aligned}$$

So, by using the last inequality and Lemma 3.2, we get

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}} \geq \|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0\to\mathcal{Z}} \succeq \max\{\limsup_{|\varphi(z)|\to1} \widetilde{A_1}(z)\log\frac{2}{1-|\varphi(z)|^2},\ \limsup_{j\to\infty} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\}.$$

Now we prove the other side. For any fix positive integer $k \geq 1$ and $i = 1, 2$, since $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ is bounded, from Theorem 2.2, we get

$$\begin{aligned}&\|T_{u,v,\varphi}f_{i,a}\|_{\mathcal{Z}} \leq C_i(1-|a|^2)^{i+2}\sum_{j=0}^{\infty}\binom{i+j}{j}|a|^j\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\\ =&(1-|a|^2)^{i+2}\left(\|u\|_{\mathcal{Z}} + \sum_{j=1}^{k-1}\binom{i+j}{j}j|a|^j j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\right) + (1-|a|^2)^{i+2}\sum_{j=k}^{\infty}\binom{i+j}{j}j|a|^j j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\\ \leq&2Q(k-1)\binom{i+k-1}{k-1}(1-|a|^k)(1-|a|^2)^{i+1} + i2^{i+2}\sup_{j\geq k} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}},\end{aligned}$$

where $Q := \max\{\sup_{j\geq1} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}, \|u\|_{\mathcal{Z}}\}$. Letting $|a| \to 1$, we obtain

$$\sigma_i = \limsup_{|a|\to1}\|T_{u,v,\varphi}f_{i,a}\|_{\mathcal{Z}} \preceq \sup_{j\geq k} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}.$$

Using last inequalitty and Theorem 3.3, we get

$$\begin{aligned}\|T_{u,v,\varphi}\|_{e,\mathcal{Z}\to\mathcal{Z}} &\preceq \max\{\limsup_{|\varphi(z)|\to1}\widetilde{A_1}(z)\log\frac{2}{1-|\varphi(z)|^2}, \sigma_1, \sigma_2\}\\ &\preceq \max\{\limsup_{|\varphi(z)|\to1}\widetilde{A_1}(z)\log\frac{2}{1-|\varphi(z)|^2},\ \limsup_{j\to\infty} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\}.\end{aligned}$$

The proof is complete □

From Theorems 3.3 and 3.4, we get the following corollary.

Corollary 3.5. *Let $u, v \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ be bounded. Then the following statements are equivalent.*

(a) The operator $T_{u,v,\varphi} : \mathcal{Z} \to \mathcal{Z}$ is compact.

(b) The operator $T_{u,v,\varphi} : \mathcal{Z}_0 \to \mathcal{Z}$ is compact.

(c)

$$\limsup_{|\varphi(z)|\to1}\widetilde{A_1}(z)\log\frac{2}{1-|\varphi(z)|^2} = \limsup_{j\to\infty} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} = 0.$$

(d)

$$\limsup_{|\varphi(z)|\to1}\widetilde{A_1}(z)\log\frac{2}{1-|\varphi(z)|^2} = \limsup_{|a|\to1}\|T_{u,v,\varphi}f_{1,a}\|_{\mathcal{Z}} = \limsup_{|a|\to1}\|T_{u,v,\varphi}f_{2,a}\|_{\mathcal{Z}} = 0.$$

(e)

$$\limsup_{|\varphi(z)|\to1}\widetilde{A_1}(z)\log\frac{2}{1-|\varphi(z)|^2} = \limsup_{|\varphi(z)|\to1}\frac{\widetilde{A_2}(z)}{(1-|\varphi(z)|^2)} = \limsup_{|\varphi(z)|\to1}\frac{\widetilde{A_3}(z)}{(1-|\varphi(z)|^2)^2} = 0.$$

Remark 3.6. *Putting $v \equiv 0$ in Theorems 2.2, 3.3, 3.4, and Corollary 3.5, we get some new characterizations for boundedness, essential norm and compactness of operator $uC_\varphi : \mathcal{Z} \to \mathcal{Z}$ (see Theorems 3.1 and 4.3 in [14] and Theorems 2.2. and 3.5 in [2]).*

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Please cite this article using:

Ebrahim Abbasi, A class of operator related weighted composition operators between Zygmund space,AUT J. Math. Com., 2(1) (2021) 17-25
DOI: 10.22060/ajmc.2020.18833.1041