



## A class of operator related weighted composition operators between Zygmund space

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**ABSTRACT:** Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$  and  $H(\mathbb{D})$  be the set of all analytic functions on  $\mathbb{D}$ . Let  $u, v \in H(\mathbb{D})$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . A class of operator related weighted composition operators is defined as follow

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

In this work, we obtain some new characterizations for boundedness and essential norm of operator  $T_{u,v,\varphi}$  between Zygmund space.

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### 1. Introduction

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$  and  $H(\mathbb{D})$  be the set of all analytic functions on  $\mathbb{D}$ . For a function  $u \in H(\mathbb{D})$  and analytic self-map  $\varphi$  of  $\mathbb{D}$  ( $\varphi(\mathbb{D}) \subset \mathbb{D}$ ), the weighted composition operator  $uC_\varphi$  on  $H(\mathbb{D})$  is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

When  $u \equiv 1$ , we get the composition operator  $C_\varphi$ , which is defined by  $C_\varphi(f) = f \circ \varphi$ . For more information about weighted composition operators see [1, 2, 13, 14].

Let  $u, v \in H(\mathbb{D})$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . S. Stević and co-authors in [9] defined the operator  $T_{u,v,\varphi}$  as follows

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let  $D$  denote the differentiation operator then  $T_{u,v,\varphi} = uC_\varphi + vC_\varphi D$ . More information about the operator  $T_{u,v,\varphi}$  can be found in [4, 7, 9, 10, 15, 16, 17]. Product-type operators on some spaces of analytic functions on the unit disk and the unit ball or the upper half-plane have become a subject of increasing interest in the last five years (see, e.g., the following representative papers [3, 5, 6, 11], and the related references therein).

A function  $f \in H(\mathbb{D})$  is said to be in the Zygmund space  $\mathcal{Z}$ , if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| < \infty.$$

The space  $\mathcal{Z}$  becomes a Banach space with the following norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| < \infty.$$

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The little Zygmund space  $\mathcal{Z}_0$ , is a closed subspace of  $\mathcal{Z}$ , consists of all function  $f \in \mathcal{Z}$  for which  $\lim_{|z| \rightarrow 1} (1 - |z|^2)|f''(z)| = 0$ .

From [18, Proposition 8], we get the next lemma.

**Lemma 1.1.** For any  $f \in \mathcal{Z}$  and  $n \in \mathbb{N}$ ,

$$\|f\|_{\mathcal{Z}} \approx \sum_{i=0}^n |f^{(i)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n+1} |f^{(n+2)}(z)|.$$

**Lemma 1.2 ([14]).** Let  $f \in \mathcal{Z}$ . Then,

$$|f(z)| \leq \|f\|_{\mathcal{Z}} \quad \text{and} \quad |f'(z)| \leq \|f\|_{\mathcal{Z}} \log \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

**Lemma 1.3 ([14]).** Let  $\{f_n\}$  be a bounded sequence in  $\mathcal{Z}$  which converges to zero uniformly on compact subsets of  $\mathbb{D}$ . Then

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{D}} |f_n(z)| = 0.$$

Recall that the essential norm of a continuous linear operator  $T : X \rightarrow Y$  is the distance from  $T$  to the compact operators, that is

$$\|T\|_{e, X \rightarrow Y} = \inf \{ \|T - K\| : K : X \rightarrow Y \text{ is compact} \}.$$

Here  $X$  and  $Y$  are Banach spaces. Notice that  $\|T\|_e = 0$  if and only if  $T$  is compact.

Recently, Liu and Yu in [7] studied the boundedness and compactness of operator  $T_{u,v,\varphi}$  from the Besov spaces into the weighted-type space  $H_{\mu}^{\infty}$ . In this work, we find some characterizations for boundedness and essential norm of operator  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$ . As some applications, we get some new characterizations of the boundedness, essential norm and compactness of weighted composition operators between Zygmund space.

Throughout this paper, we say that  $A \succeq B$ , if there exists a constant  $C$  such that  $A \geq CB$ . The symbol  $A \approx B$  means that  $A \succeq B \succeq A$ .

## 2. Boundedness

In this section, the boundedness of operator  $T_{u,v,\varphi}$  between Zygmund spaces is characterized. We begin with the next lemma.

**Lemma 2.1.** Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then for any  $a \in \mathbb{D}$ , there exists a function  $\Psi_a$  in  $\mathcal{Z}_0$  such that  $\sup_{a \in \mathbb{D}} \|\Psi_a\|_{\mathcal{Z}} < \infty$  and

$$\Psi_a(\varphi(a)) = \Psi_a''(\varphi(a)) = \Psi_a'''(\varphi(a)) = 0 \quad \text{and} \quad \Psi_a'(\varphi(a)) = \log \frac{2}{1 - |\varphi(a)|^2}.$$

**Proof.** If  $\varphi(a) = 0$ , then set  $\Psi_a(z) = \int_0^z \log 2 \, d\xi$ , as desired. For any  $a \in \mathbb{D}$  with  $\varphi(a) \neq 0$  and  $k \in \{1, 2, 3\}$ , set

$$h_{a,k}(z) = \frac{(k+3)!}{k!} + \int_{\varphi(a)}^z \left( (3+k) \log \frac{2}{1 - \varphi(a)\xi} - \frac{(\log \frac{2}{1 - \varphi(a)\xi})^{3+k}}{(\log \frac{2}{1 - |\varphi(a)|^2})^{2+k}} \right) d\xi.$$

It is obvious that  $h_{a,k} \in \mathcal{Z}_0$ . In this case

$$\Psi_a(z) = 5h_{a,1}(z) - 6h_{a,2}(z) + 2h_{a,3}(z)$$

as desired. By simple calculation, we get

$$\sup_{a \in \mathbb{D}} \|h_{a,k}\|_{\mathcal{Z}} < \infty \quad k \in \{1, 2, 3\}.$$

Hence,  $\sup_{a \in \mathbb{D}} \|\Psi_a\|_{\mathcal{Z}} < \infty$ .

□

For simplicity in calculation, we set

$$\begin{aligned} \widetilde{A}_0(z) &= (1 - |z|^2)|u''(z)|, & \widetilde{A}_1(z) &= (1 - |z|^2)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)| \\ \widetilde{A}_2(z) &= (1 - |z|^2)|u(z)\varphi'^2(z) + 2v'(z)\varphi'(z) + v(z)\varphi''(z)|, & \widetilde{A}_3(z) &= (1 - |z|^2)|v(z)\varphi'^2(z)|. \end{aligned} \quad (1)$$

**Theorem 2.2.** *Let  $u \in \mathcal{Z}$ ,  $v \in H(\mathbb{D})$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then the following statements are equivalent.*

(a) *The operator  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  is bounded.*

(b) *The operator  $T_{u,v,\varphi} : \mathcal{Z}_0 \rightarrow \mathcal{Z}$  is bounded.*

(c)  $\max \left\{ \sup_{z \in \mathbb{D}} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, \sup_{j \geq 1} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} \right\} < \infty$ .

(d)  $\max \left\{ \sup_{z \in \mathbb{D}} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi} f_{1,a}\|_{\mathcal{Z}} \right\} < \infty$  and

$$\max \left\{ \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi} f_{2,a}\|_{\mathcal{Z}} < \infty, \sup_{z \in \mathbb{D}} \widetilde{A}_2(z), \sup_{z \in \mathbb{D}} \widetilde{A}_3(z) \right\} < \infty,$$

where  $f_{i,a}(z) = \frac{(1 - |a|^2)^{i+2}}{(1 - \bar{a}z)^{i+1}}$ .

(e)

$$\max \left\{ \sup_{z \in \mathbb{D}} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, \sup_{z \in \mathbb{D}} \frac{\widetilde{A}_2(z)}{(1 - |\varphi(z)|^2)}, \sup_{z \in \mathbb{D}} \frac{\widetilde{A}_3(z)}{(1 - |\varphi(z)|^2)^2} \right\} < \infty.$$

**Proof.** (a)  $\Rightarrow$  (b) It is obvious.

(b)  $\Rightarrow$  (c) For any  $a \in \mathbb{D}$ , let  $\Psi_a$  be the function defined in Lemma 2.1.

$$\begin{aligned} \widetilde{A}_1(a) \log \frac{2}{1 - |\varphi(a)|^2} &= \widetilde{A}_1(a) |\Psi'_a(\varphi(a))| = (1 - |a|^2) |(T_{u,v,\varphi} \Psi_a)''(a)| \leq \|T_{u,v,\varphi} \Psi_a\|_{\mathcal{Z}} \\ &\leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} \sup_{a \in \mathbb{D}} \|\Psi_a\|_{\mathcal{Z}} < \infty. \end{aligned}$$

From [8], we know that the sequence  $\{z^j\}_0^\infty$  is bounded in  $\mathcal{B}_0$ , hence  $\{j^{-1}z^j\}_1^\infty$  is bounded in  $\mathcal{Z}_0$ , therefore

$$\sup_{j \geq 1} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} \sup_{j \geq 1} \|z^j\|_{\mathcal{B}} < \infty.$$

(c)  $\Rightarrow$  (d) Let  $p_j(z) = z^j$ . For any  $a \in \mathbb{D}$  and  $i = 1, 2$ , we have

$$\begin{aligned} \|T_{u,v,\varphi} f_{i,a}\|_{\mathcal{Z}} &\leq (1 - |a|^2)^{i+1} \left( \|u\|_{\mathcal{Z}} + \sum_{j=1}^\infty \binom{i+j-1}{j} |a|^j j^{-1} \|T_{u,v,\varphi} p_j\|_{\mathcal{W}_\mu^n} \right) \\ &\leq (1 + i2^{i+1}) \max\{\|u\|_{\mathcal{Z}}, \sup_{j \geq 1} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\}. \end{aligned}$$

Since  $a$  is arbitrary,  $\sup_{a \in \mathbb{D}} \|T_{u,v,\varphi} f_{i,a}\|_{\mathcal{Z}} < \infty$ . Applying the operator  $T_{u,v,\varphi}$  for  $p_2(z) = z^2$ , so

$$\widetilde{A}_2(z) \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} + \widetilde{A}_0(z) |\varphi(z)|^2 + 2\widetilde{A}_1(z) |\varphi(z)| \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} + \sup_{z \in \mathbb{D}} \widetilde{A}_0(z) + 2 \sup_{z \in \mathbb{D}} \widetilde{A}_1(z) < \infty.$$

Therefore,  $\sup_{z \in \mathbb{D}} \widetilde{A}_2(z) < \infty$ . Similarly by applying the operator  $T_{u,v,\varphi}$  for  $p_3(z) = z^3$ , we get

$$\widetilde{A}_3(z) \leq \|T_{u,v,\varphi}\|_{\mathcal{Z}} + \sup_{z \in \mathbb{D}} \widetilde{A}_0(z) + 3 \sup_{z \in \mathbb{D}} \widetilde{A}_1(z) + 6 \sup_{z \in \mathbb{D}} \widetilde{A}_2(z) < \infty.$$

Hence  $\sup_{z \in \mathbb{D}} \widetilde{A}_3(z) < \infty$ .

(d)  $\Rightarrow$  (e) Suppose that (d) holds. we set

$$k_{1,a}(z) = \frac{5}{6} f_{1,a}(z) - \frac{1}{3} f_{2,a}(z). \quad (2)$$

Let  $|\varphi(a)| > \frac{1}{2}$ , so

$$\begin{aligned} \frac{\widetilde{A}_2(a)|\varphi(a)|^2}{(1-|\varphi(a)|^2)} &\leq \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}k_{1,\varphi(a)}\|_{\mathcal{Z}} + \sup_{a \in \mathbb{D}} \widetilde{A}_0(a)(1-|\varphi(a)|^2) + \sup_{a \in \mathbb{D}} \widetilde{A}_1(a)|\varphi(a)| \\ &\preceq \sup_{a \in \mathbb{D}} \|f_{1,a}\|_{\mathcal{Z}} + \sup_{a \in \mathbb{D}} \|f_{2,a}\|_{\mathcal{Z}} + \sup_{a \in \mathbb{D}} \widetilde{A}_0(a) + \sup_{a \in \mathbb{D}} \widetilde{A}_1(a) < \infty. \end{aligned}$$

From previous inequality,  $\sup_{|\varphi(a)| > \frac{1}{2}} \frac{\widetilde{A}_2(a)}{(1-|\varphi(a)|^2)} < \infty$ . Also by using (d), we obtain

$$\sup_{|\varphi(a)| \leq \frac{1}{2}} \frac{\widetilde{A}_2(a)}{(1-|\varphi(a)|^2)} \leq \frac{4}{3} \sup_{|\varphi(a)| \leq \frac{1}{2}} \widetilde{A}_2(a) < \infty.$$

Now we set

$$k_{2,a}(z) = \frac{-1}{6} f_{1,a}(z) + \frac{1}{12} f_{2,a}(z). \tag{3}$$

Let  $|\varphi(a)| > \frac{1}{2}$ , hence

$$\begin{aligned} \frac{\widetilde{A}_3(a)|\varphi(a)|^3}{(1-|\varphi(a)|^2)^2} &\leq \sup_{a \in \mathbb{D}} \|T_{u,v,\varphi}k_{2,\varphi(a)}\|_{\mathcal{Z}} + \sup_{a \in \mathbb{D}} \widetilde{A}_0(a)(1-|\varphi(a)|^2) + \sup_{a \in \mathbb{D}} \widetilde{A}_1(a)|\varphi(a)| \\ &\preceq \sup_{a \in \mathbb{D}} \|f_{1,a}\|_{\mathcal{Z}} + \sup_{a \in \mathbb{D}} \|f_{2,a}\|_{\mathcal{Z}} + \sup_{a \in \mathbb{D}} \widetilde{A}_0(a) + \sup_{a \in \mathbb{D}} \widetilde{A}_1(a) < \infty. \end{aligned}$$

So,  $\sup_{|\varphi(a)| > \frac{1}{2}} \frac{\widetilde{A}_3(a)}{(1-|\varphi(a)|^2)^2} < \infty$ . Also from (d), we have

$$\sup_{|\varphi(a)| \leq \frac{1}{2}} \frac{\widetilde{A}_3(a)}{(1-|\varphi(a)|^2)^2} \leq \frac{16}{9} \sup_{|\varphi(a)| \leq \frac{1}{2}} \widetilde{A}_3(a) < \infty.$$

(e)  $\Rightarrow$  (a) Let  $f$  be arbitrary function in  $\mathcal{Z}$ . Using Lemmas 1.1 and 1.2, we have

$$\begin{aligned} (1-|z|^2)|(T_{u,v,\varphi}f)''(z)| &\preceq \|f\|_{\mathcal{Z}}\|u\|_{\mathcal{Z}} + \|f\|_{\mathcal{Z}} \sup_{z \in \mathbb{D}} \widetilde{A}_1(z) \log \frac{2}{1-|\varphi(z)|^2} \\ &\quad + \|f\|_{\mathcal{Z}} \sup_{z \in \mathbb{D}} \frac{\widetilde{A}_2(z)}{(1-|\varphi(z)|^2)} + \|f\|_{\mathcal{Z}} \sup_{z \in \mathbb{D}} \frac{\widetilde{A}_3(z)}{(1-|\varphi(z)|^2)^2}. \end{aligned}$$

Also

$$|(T_{u,v,\varphi}f)(0)| \leq \|f\|_{\mathcal{Z}} \left( |u(0)| + |v(0)| \log \frac{2}{1-|\varphi(0)|^2} \right)$$

and

$$|(T_{u,v,\varphi}f)'(0)| \leq \|f\|_{\mathcal{Z}} \left( |u'(0)| + |u(0)\varphi'(0) + v'(0)| \log \frac{2}{1-|\varphi(0)|^2} + \frac{|v(0)|}{1-|\varphi(0)|^2} \right)$$

Thus,  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  is bounded. The proof is completed. □

### 3. Essential norm

In this section, some estimates for the essential norm of operator  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  are obtained. For the study of the essential norm, we need the following lemma, which can be proved in a standard way, see, for example [12, Lemma 2.10].

**Lemma 3.1.** *Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $S : \mathcal{Z}(\mathcal{Z}_0) \rightarrow \mathcal{Z}$  be bounded. Then  $S$  is compact if and only if whenever  $\{f_k\}$  is bounded in  $\mathcal{Z}(\mathcal{Z}_0)$  and  $f_k \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ , then*

$$\lim_{k \rightarrow \infty} \|Sf_k\|_{\mathcal{Z}} = 0.$$

**Lemma 3.2.** Let  $u, v \in H(\mathbb{D})$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ , such that  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  be bounded. Then

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0 \rightarrow \mathcal{Z}} \geq \limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}.$$

where  $\widetilde{A}_1(z)$  is defined in (1).

**Proof.** Let  $\{z_i\}$  be a sequence in  $\mathbb{D}$  such that  $\lim |\varphi(z_i)| = 1$ . We assume that for each  $i$ ,  $\varphi(z_i) \neq 0$ . First we show that there exists a bounded sequence  $\{\Psi_i\}$  in  $\mathcal{Z}_0$  such that,  $\{\Psi_i\}$  converge to 0 uniformly on compact subsets of  $\mathbb{D}$  and

$$\Psi_i(\varphi(z_i)) = \Psi_i''(\varphi(z_i)) = \Psi_i'''(\varphi(z_i)) = 0, \quad \Psi_i'(\varphi(z_i)) = \log \frac{2}{1 - |\varphi(z_i)|^2}.$$

For each  $i$  and  $k \in \{1, 2, 3\}$ , we set

$$h_{i,k}(z) = (k+1)(k+2)(k+3) + \int_{\varphi(z_i)}^z \left( (3+k) \frac{\left(\log \frac{2}{1 - |\varphi(z_i)\xi|^2}\right)^{2+k}}{\left(\log \frac{2}{1 - |\varphi(z_i)|^2}\right)^{1+k}} - (2+k) \frac{\left(\log \frac{2}{1 - |\varphi(z_i)\xi|^2}\right)^{3+k}}{\left(\log \frac{2}{1 - |\varphi(z_i)|^2}\right)^{2+k}} \right) d\xi.$$

It is clear that  $h_{i,k} \in \mathcal{Z}_0$ . Now the following sequence give us all mentioned properties

$$\Psi_i(z) = \frac{1}{12}h_{i,1}(z) - \frac{3}{20}h_{i,2}(z) + \frac{1}{15}h_{i,3}(z).$$

Let  $K : \mathcal{Z}_0 \rightarrow \mathcal{Z}$  be arbitrary compact operator. By using Lemma 3.1, we have

$$\begin{aligned} \|(T_{u,v,\varphi} - K)\Psi_i\|_{\mathcal{Z} \rightarrow \mathcal{Z}} &\geq \|(T_{u,v,\varphi} - K)\Psi_i\|_{\mathcal{Z}_0 \rightarrow \mathcal{Z}} \geq \limsup_{i \rightarrow \infty} \|T_{u,v,\varphi}\Psi_i\|_{\mathcal{Z}} - \limsup_{i \rightarrow \infty} \|K\Psi_i\|_{\mathcal{Z}} \\ &\geq \limsup_{i \rightarrow \infty} \widetilde{A}_1(z_i) \log \frac{2}{1 - |\varphi(z_i)|^2} = \limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}. \end{aligned}$$

Based on the definition of essential norm, we have

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0 \rightarrow \mathcal{Z}} = \inf_K \|T_{u,v,\varphi} - K\| \geq \limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}.$$

The proof is completed. □

**Theorem 3.3.** Let  $u, v \in H(\mathbb{D})$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  be bounded. Then

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}} \approx \max\{\rho_i\}_{i=0}^2 \approx \max\{\sigma_i\}_{i=0}^2$$

where

$$\begin{aligned} f_{i,a}(z) &= \frac{(1 - |a|^2)^{i+2}}{(1 - \bar{a}z)^{i+1}}, & \sigma_0 = \rho_0 &= \limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, & \sigma_1 &= \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}f_{1,a}\|_{\mathcal{Z}}, \\ \sigma_2 &= \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}f_{2,a}\|_{\mathcal{Z}}, & \rho_1 &= \limsup_{|\varphi(z)| \rightarrow 1} \frac{\widetilde{A}_2(z)}{(1 - |\varphi(z)|^2)}, & \rho_2 &= \limsup_{|\varphi(z)| \rightarrow 1} \frac{\widetilde{A}_3(z)}{(1 - |\varphi(z)|^2)^2} \end{aligned}$$

and  $\widetilde{A}_0(z), \widetilde{A}_1(z), \widetilde{A}_2(z), \widetilde{A}_3(z)$  are defined in (1).

**Proof.** Let  $\{z_j\}_{j \in \mathbb{N}}$  be a sequence in  $\mathbb{D}$  such that  $\lim |\varphi(z_j)| = 1$  and  $k_{i,a} (i = 1, 2)$  be functions are defined in (2) and (3). It is clear that for all  $a \in \mathbb{D}$ ,  $\|k_{i,a}\|_{\mathcal{Z}} \leq 1 (i = 1, 2)$  and if  $a \neq 0$  then  $k_{i,a} \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$  as  $|a| \rightarrow 1$ . So, by using Lemma 3.1 for any compact operator  $K$  from  $\mathcal{Z}$  into  $\mathcal{Z}$ , we get

$$\begin{aligned} \|T_{u,v,\varphi} - K\|_{\mathcal{Z} \rightarrow \mathcal{Z}} &\geq \limsup_{j \rightarrow \infty} \|(T_{u,v,\varphi} - K)k_{i,\varphi(z_j)}\|_{\mathcal{Z}} \geq \limsup_{j \rightarrow \infty} \|T_{u,v,\varphi}k_{i,\varphi(z_j)}\|_{\mathcal{Z}} - \limsup_{j \rightarrow \infty} \|Kk_{i,\varphi(z_j)}\|_{\mathcal{Z}} \\ &\geq \limsup_{j \rightarrow \infty} \frac{\widetilde{A}_{i+1}(z_j)|\varphi(z_j)|^2}{(1 - |\varphi(z_j)|^2)^i} - \limsup_{j \rightarrow \infty} \widetilde{A}_0(z_j)(1 - |\varphi(z_j)|^2) - \limsup_{j \rightarrow \infty} \widetilde{A}_1(z_j)|\varphi(z_j)|. \end{aligned}$$

From Theorem 2.2(c), we have  $\limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) = 0$ , so from previous inequality and definition of essential norm, we have

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}} = \inf_K \|T_{u,v,\varphi} - K\|_{\mathcal{Z} \rightarrow \mathcal{Z}} \succeq \max\{\rho_1, \rho_2\}.$$

Using Lemma 3.2, we get

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}} \succeq \max\{\rho_0, \rho_1, \rho_2\}.$$

Also  $\|f_{i,a}\|_{\mathcal{Z}} \leq 1 (i = 1, 2)$  and  $f_{i,a} \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$  as  $|a| \rightarrow 1$ . So, for any compact operator  $K : \mathcal{Z} \rightarrow \mathcal{Z}$ , by Lemma 3.1, we get  $\limsup_{|a| \rightarrow 1} \|K f_{i,a}\|_{\mathcal{Z}} = 0$ . Hence

$$\|T_{u,v,\varphi} - K\|_{\mathcal{Z} \rightarrow \mathcal{Z}} \succeq \limsup_{|a| \rightarrow 1} \|(T_{u,v,\varphi} - K)f_{i,a}\|_{\mathcal{Z}} \geq \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi} f_{i,a}\|_{\mathcal{Z}} - \limsup_{|a| \rightarrow 1} \|K f_{i,a}\|_{\mathcal{Z}} = \sigma_i.$$

By the last inequality and Lemma 3.2, we obtain

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}} = \inf_K \|T_{u,v,\varphi} - K\|_{\mathcal{Z} \rightarrow \mathcal{Z}} \succeq \max\{\sigma_0, \sigma_1, \sigma_2\}.$$

Now, we prove that

$$\min\{\max\{\sigma_i\}_{i=0}^2, \max\{\rho_i\}_{i=0}^2\} \succeq \|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}}.$$

For  $r \in [0, 1)$ , we define  $K_r f(z) = f_r(z) = f(rz)$ . It is clear that  $K_r : \mathcal{Z} \rightarrow \mathcal{Z}$  is a compact operator with  $\|K_r\| \leq 1$ . Also we know that  $f_r \rightarrow f$  uniformly on compact subsets of  $\mathbb{D}$  as  $r \rightarrow 1$ . Let  $\{r_j\} \subset (0, 1)$  be a sequence such that  $r_j \rightarrow 1$  as  $j \rightarrow \infty$ . Then for any positive integer  $j$ , the operator  $T_{u,v,\varphi} K_{r_j} : \mathcal{Z} \rightarrow \mathcal{Z}$  is compact. So

$$\limsup_{j \rightarrow \infty} \|T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_j}\| \geq \|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}}.$$

Therefore, based on the definition of essential norm it is enough to prove that

$$\min\{\max\{\sigma_i\}_{i=0}^2, \max\{\rho_i\}_{i=0}^2\} \succeq \limsup_{j \rightarrow \infty} \|T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_j}\|_{\mathcal{Z} \rightarrow \mathcal{Z}}.$$

Let  $f$  be arbitrary function in  $\mathcal{Z}$  such that  $\|f\|_{\mathcal{Z}} \leq 1$ ,

$$\begin{aligned} \|(T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_j})f\|_{\mathcal{Z}} &\leq \underbrace{(|u(0)| + |u'(0)|)(|f - f_{r_j}(\varphi(0))|)}_{R_1} + \underbrace{|v(0)\varphi'(0)|(|f - f_{r_j}(\varphi(0))'|)}_{R_2} \\ &+ \underbrace{(|v(0)| + |u(0)\varphi'(0)| + |v'(0)|)(|f - f_{r_j}(\varphi(0))'|)}_{R_3} + \underbrace{\sup_{z \in \mathbb{D}} \widetilde{A}_0(z)|f - f_{r_j}(\varphi(z))|}_{L_0} + \underbrace{\sup_{|\varphi(z)| \leq r_N} \widetilde{A}_1(z)|f - f_{r_j}(\varphi(z))'|}_{L_{11}} \\ &+ \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_1(z)|f - f_{r_j}(\varphi(z))'|}_{L_{12}} + \underbrace{\sup_{|\varphi(z)| \leq r_N} \widetilde{A}_2(z)|f - f_{r_j}(\varphi(z))''|}_{L_{21}} + \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_2(z)|f - f_{r_j}(\varphi(z))''|}_{L_{22}} \\ &+ \underbrace{\sup_{|\varphi(z)| \leq r_N} \widetilde{A}_3(z)|f - f_{r_j}(\varphi(z))'''|}_{L_{31}} + \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_3(z)|f - f_{r_j}(\varphi(z))'''|}_{L_{32}} \end{aligned} \tag{4}$$

where  $N \in \mathbb{N}$  and  $r_j \geq \frac{1}{2}$  for all  $j \geq N$ . Since for any  $k \in \mathbb{N}_0$ ,  $(f - f_{r_j})^{(k)} \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$  as  $j \rightarrow \infty$ , from Lemmas 1.2, 1.3 and Theorem 2.2 (d), we get

$$\limsup_{j \rightarrow \infty} R_t = \limsup_{j \rightarrow \infty} L_0 = \limsup_{j \rightarrow \infty} L_{t1} = 0 \quad (t = 1, 2, 3). \tag{5}$$

Also

$$L_{12} \leq \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_1(z)|f'(\varphi(z))|}_{L_{12}^1} + \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_1(z)|r_j f'(r_j \varphi(z))|}_{L_{12}^2}. \tag{6}$$

No we obtain estimate for  $L_{12}^1$ . Using Lemma 1.2,

$$L_{12}^1 = \sup_{|\varphi(z)| > r_N} \widetilde{A}_1(z) |f'(\varphi(z))| \leq \sup_{|\varphi(z)| > r_N} \widetilde{A}_1(z) \|f\|_{\mathcal{Z}} \log \frac{2}{1 - |\varphi(z)|^2}.$$

Letting  $N \rightarrow \infty$ , we get

$$\limsup_{j \rightarrow \infty} L_{12}^1 \preceq \sigma_0 = \rho_0. \tag{7}$$

Similarly, we have

$$\limsup_{j \rightarrow \infty} L_{12}^2 \preceq \sigma_0 = \rho_0. \tag{8}$$

On the other hand

$$\begin{aligned} L_{22} &\leq \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_2(z) |f''(\varphi(z))|}_{L_{22}^1} + \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_2(z) |r_j f''(r_j \varphi(z))|}_{L_{22}^2}, \\ L_{32} &\leq \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_3(z) |f'''(\varphi(z))|}_{L_{32}^1} + \underbrace{\sup_{|\varphi(z)| > r_N} \widetilde{A}_3(z) |r_j f'''(r_j \varphi(z))|}_{L_{32}^2}. \end{aligned} \tag{9}$$

Now we estimate  $L_{2s}^1$  ( $s = 2, 3$ ). From Lemmas 1.1, 1.2, and (2) and (3),

$$\begin{aligned} L_{s2}^1 &= \sup_{|\varphi(z)| > r_N} \frac{(1 - |\varphi(z)|^2)^{s-1} |f^{(s)}(\varphi(z))|}{|\varphi(z)|^s} \frac{|\varphi(z)|^s \widetilde{A}_s(z)}{(1 - |\varphi(z)|^2)^{s-1}} \preceq \|f\|_{\mathcal{Z}} \sup_{|\varphi(z)| > r_N} \|T_{u,v,\varphi} k_{s,\varphi(z)}\|_{\mathcal{Z}} \\ &\preceq \sup_{|a| > r_N} \|T_{u,v,\varphi} f_{1,a}\|_{\mathcal{Z}} + \sup_{|a| > r_N} \|T_{u,v,\varphi} f_{2,a}\|_{\mathcal{Z}}. \end{aligned}$$

Letting  $N \rightarrow \infty$ , we get

$$\limsup_{j \rightarrow \infty} L_{s2}^1 \preceq \rho_{s-1} \quad \text{and} \quad \limsup_{j \rightarrow \infty} L_{s2}^2 \preceq \max\{\sigma_1, \sigma_2\}. \tag{10}$$

Similarly, we have

$$\limsup_{j \rightarrow \infty} L_{s2}^2 \preceq \rho_{s-1} \quad \text{and} \quad \limsup_{j \rightarrow \infty} L_{s2}^2 \preceq \max\{\sigma_1, \sigma_2\}. \tag{11}$$

By using (4), (5), (6), (7), (8), (9), (10) and (11), we obtain

$$\max\{\sigma_0, \sigma_1, \sigma_2\} \succeq \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{Z}} \leq 1} \|(T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_j})f\|_{\mathcal{Z}} = \limsup_{j \rightarrow \infty} \|T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_j}\|_{\mathcal{Z} \rightarrow \mathcal{Z}}$$

and

$$\max\{\rho_0, \rho_1, \rho_2\} \succeq \limsup_{j \rightarrow \infty} \|T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_j}\|_{\mathcal{Z} \rightarrow \mathcal{Z}}.$$

Hence,

$$\min\{\max\{\sigma_i\}_{i=0}^2, \max\{\rho_i\}_{i=0}^2\} \succeq \limsup_{j \rightarrow \infty} \|T_{u,v,\varphi} - T_{u,v,\varphi} K_{r_j}\|_{\mathcal{Z} \rightarrow \mathcal{Z}}.$$

The proof is completed. □

**Theorem 3.4.** Let  $u, v \in H(\mathbb{D})$ ,  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  be bounded. Then

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}} \approx \|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0 \rightarrow \mathcal{Z}} \approx \max\left\{ \limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, \limsup_{j \rightarrow \infty} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} \right\}$$

where  $\widetilde{A}_1(z)$  is defined in (1).

**Proof.** Let  $p_j(z) = z^j$ . It is clear that  $\{j^{-1}p_j\}_1^\infty \subset \mathcal{Z}_0$  and  $j^{-1}p_j \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$  as  $j \rightarrow \infty$ . So, for any compact operator  $K : \mathcal{Z}_0 \rightarrow \mathcal{Z}$ , we have  $\lim_{j \rightarrow \infty} j^{-1}\|Kp_j\|_{\mathcal{Z}} = 0$ . Thus,

$$\begin{aligned} \|T_{u,v,\varphi} - K\|_{\mathcal{Z}_0 \rightarrow \mathcal{Z}} &\succeq \limsup_{j \rightarrow \infty} j^{-1}\|(T_{u,v,\varphi} - K)p_j\|_{\mathcal{Z}} \limsup_{j \rightarrow \infty} j^{-1}\|T_{u,v,\varphi}p_j\|_{\mathcal{Z}} - \limsup_{j \rightarrow \infty} j^{-1}\|Kp_j\|_{\mathcal{Z}} \\ &= \limsup_{j \rightarrow \infty} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}. \end{aligned}$$

So, by using the last inequality and Lemma 3.2, we get

$$\|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}} \geq \|T_{u,v,\varphi}\|_{e,\mathcal{Z}_0 \rightarrow \mathcal{Z}} \succeq \max\{\limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, \limsup_{j \rightarrow \infty} j^{-1}\|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\}.$$

Now we prove the other side. For any fix positive integer  $k \geq 1$  and  $i = 1, 2$ , since  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  is bounded, from Theorem 2.2, we get

$$\begin{aligned} \|T_{u,v,\varphi}f_{i,a}\|_{\mathcal{Z}} &\leq C_i(1 - |a|^2)^{i+2} \sum_{j=0}^\infty \binom{i+j}{j} |a|^j \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} \\ &= (1 - |a|^2)^{i+2} \left( \|u\|_{\mathcal{Z}} + \sum_{j=1}^{k-1} \binom{i+j}{j} j |a|^j j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} \right) + (1 - |a|^2)^{i+2} \sum_{j=k}^\infty \binom{i+j}{j} j |a|^j j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} \\ &\leq 2Q(k-1) \binom{i+k-1}{k-1} (1 - |a|^k)(1 - |a|^2)^{i+1} + i2^{i+2} \sup_{j \geq k} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}, \end{aligned}$$

where  $Q := \max\{\sup_{j \geq 1} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}, \|u\|_{\mathcal{Z}}\}$ . Letting  $|a| \rightarrow 1$ , we obtain

$$\sigma_i = \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}f_{i,a}\|_{\mathcal{Z}} \preceq \sup_{j \geq k} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}.$$

Using last inequality and Theorem 3.3, we get

$$\begin{aligned} \|T_{u,v,\varphi}\|_{e,\mathcal{Z} \rightarrow \mathcal{Z}} &\preceq \max\{\limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, \sigma_1, \sigma_2\} \\ &\preceq \max\{\limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2}, \limsup_{j \rightarrow \infty} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}}\}. \end{aligned}$$

The proof is complete □

From Theorems 3.3 and 3.4, we get the following corollary.

**Corollary 3.5.** *Let  $u, v \in H(\mathbb{D})$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$  such that  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  be bounded. Then the following statements are equivalent.*

- (a) *The operator  $T_{u,v,\varphi} : \mathcal{Z} \rightarrow \mathcal{Z}$  is compact.*
- (b) *The operator  $T_{u,v,\varphi} : \mathcal{Z}_0 \rightarrow \mathcal{Z}$  is compact.*
- (c)

$$\limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2} = \limsup_{j \rightarrow \infty} j^{-1} \|u\varphi^j + jv\varphi^{j-1}\|_{\mathcal{Z}} = 0.$$

(d)

$$\limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2} = \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}f_{1,a}\|_{\mathcal{Z}} = \limsup_{|a| \rightarrow 1} \|T_{u,v,\varphi}f_{2,a}\|_{\mathcal{Z}} = 0.$$

(e)

$$\limsup_{|\varphi(z)| \rightarrow 1} \widetilde{A}_1(z) \log \frac{2}{1 - |\varphi(z)|^2} = \limsup_{|\varphi(z)| \rightarrow 1} \frac{\widetilde{A}_2(z)}{(1 - |\varphi(z)|^2)} = \limsup_{|\varphi(z)| \rightarrow 1} \frac{\widetilde{A}_3(z)}{(1 - |\varphi(z)|^2)^2} = 0.$$

**Remark 3.6.** *Putting  $v \equiv 0$  in Theorems 2.2, 3.3, 3.4, and Corollary 3.5, we get some new characterizations for boundedness, essential norm and compactness of operator  $uC_\varphi : \mathcal{Z} \rightarrow \mathcal{Z}$  (see Theorems 3.1 and 4.3 in [14] and Theorems 2.2. and 3.5 in [2]).*



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