Recognition by degree prime-power graph and order of some characteristically simple groups

Afsane Bahri\textsuperscript{a}, Behrooz Khosravi\textsuperscript{a}, Morteza Baniasad Azad\textsuperscript{a}

\textsuperscript{a}Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran

\textbf{ABSTRACT:} In this paper, by the order of a group and triviality of $O_p(G)$ for some prime $p$, we give a new characterization for some characteristically simple groups. In fact, we prove that if $p \in \{5, 17, 23, 37, 47, 73\}$ and $n \leq p$, where $n$ is a natural number, then $G \cong \text{PSL}(2, p)^n$ if and only if $|G| = |\text{PSL}(2, p)|^n$ and $O_p(G) = 1$.

Recently in [Qin, Yan, Shum and Chen, Comm. Algebra, 2019], the degree prime-power graph of a finite group have been introduced and it is proved that the Mathieu groups are uniquely determined by their degree prime-power graphs and orders. As a consequence of our results, we show that PSL$(2, p)^n$, where $p \in \{5, 17, 23, 37, 47, 73\}$ and $n \leq p$ are uniquely determined by their degree prime-power graphs and orders.

\textbf{Keywords:}
Degree prime power graph
Order
Characteristically simple group
Characterization

\section{1. Introduction}

Throughout the paper, $G$ is a finite group. The set of all prime divisors of $|G|$ is denoted by $\pi(G)$. By $G^n$, we mean the direct product of $n$ copies of $G$. Let $\text{cd}(G)$ be the set of irreducible character degrees of $G$, and $\rho(G)$ the set of primes dividing the elements in $\text{cd}(G)$. Some graphs are defined concerning to the irreducible characters of a finite group. We refer the readers to a survey by Lewis [7] for results concerning graphs associated with character degrees. The character degree graph of $G$, which is denoted by $\Delta(G)$ was introduced by Manz et al. in 1998 (see [8]). The vertex set of $\Delta(G)$ is $\rho(G)$ and there exists an edge between two distinct elements $a, b \in \rho(G)$, if $ab$ divides some irreducible character degree in $\text{cd}(G)$.

In [4, 5, 6], the authors proved that $A_5$, $A_5 \times A_5$ and some other groups are characterizable by their character degree graphs and orders. Obviously, $A_5^n$, where $n > 2$, is not uniquely determined by $\Delta(A_5^n)$ and $|A_5^n|$, since $A_5$ and $A_5 \times A_5 \times L$, where $L$ is a group of order $60^{n-2}$ have the same order and character degree graph (see Figure 1).

There also exist some simple groups, say $M_{12}$, which are not characterizable by their character degree graphs and orders. For this reason, Qin \textit{et al.} in [10] have introduced a new graph related to irreducible characters of a finite group as follows and they showed that the Mathieu groups are characterizable by this graph and order.

\textbf{Notation.} Let $n$ and $a$ be integers such that $(a, n) = 1$. Then, $\text{ord}_m(a)$ denotes the smallest positive integer $c$ such that $a^c \equiv 1 \pmod{m}$. In addition, for a prime $p$, we write $p^k|n$, whenever $p^k \mid n$ but $p^{k+1} \nmid n$. In this case, we also write $n_p = p^k$.

\textbf{Definition 1.1.} The degree prime-power graph $\Gamma(G)$ is defined as follow:
For each $p \in \rho(G)$ let $b(p) = \text{max}\{a_p | a \in \text{cd}(G)\}$. The vertex set of $\Gamma(G)$ is $V = \{b(p) | p \in \rho(G)\}$ and there is an edge between distinct vertices $x, y \in V$ if $xy$ divides an element of $\text{cd}(G)$.
In this paper, for a characteristically simple group $G$ we present a new characterization based on the order of $G$ and $O_p(G)$, for some $p \in \pi(G)$. Our main result is:

**Theorem 1.2.** Let $G$ be a finite group, and $M = \text{PSL}(2, p^n)$, where $p \in \{5, 17, 23, 37, 47, 73\}$, $n \leq p$. Then $G \cong M$ if and only if $|G| = |M|$ and $|O_p(G)| = |O_p(M)| = 1$.

As a consequence of our results, we show that these groups are uniquely determined by their degree prime-power graphs and orders.

2. Preliminary Results

**Lemma 2.1.** [11, Lemma] Let $G$ be a non-solvable group. Then $G$ has a normal series $1 \leq H \leq K \leq G$ such that $K/H$ is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\text{Out}(K/H)|$.

**Lemma 2.2.** [9, Theorems 3.6] Let $p$ be an odd prime, and let $a \neq \pm 1$ be an integer not divisible by $p$. Let also $d$ be the order of $a$ modulo $p$ and $k_0$ the largest integer such that $a^d \equiv 1 \pmod{p^{k_0}}$. Then the order of $a$ modulo $p^k$ is $d$ for $k = 1, \ldots, k_0$ and $dp^{k-k_0}$ for $k > k_0$.

**Lemma 2.3.** [1, Lemma 2.2] Let $S$ be a finite non-abelian simple group, and $p_0$ be the largest prime divisor of $|S|$. If $G$ is an extension of $S^m$ by $S^n$, where $m + n \leq p_0$, then $G \cong S^{m+n}$.

**Lemma 2.4.** Let $S$ be a subnormal subgroup of $G$, and $O_p(G) = 1$, where $p \in \pi(G)$. Then $O_p(S) = 1$.

**Proof.** It is straightforward.

3. Main Results

**Lemma 3.1.** Let $p \in \{5, 17, 23, 37, 47, 73\}$, $|G|$ be a divisor of $|\text{PSL}(2, p)|^n$ and $p^n \mid |G|$, where $n$ is a natural number. If $O_p(G) = 1$, then $G$ is non-solvable.

**Proof.** Let $|G| = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}p^n$ and define

$$
\beta(G) = \left(\frac{\alpha_1}{\text{ord}_{p}(p_1)} + \frac{\alpha_2}{\text{ord}_{p}(p_2)} + \cdots + \frac{\alpha_k-1}{\text{ord}_{p}(p_k-1)}\right) \frac{p}{p-1}.
$$

Firstly, in each case we prove that $\beta(G) < n$. Assume that $p = 73$. Thus, $|G| = 2^{\alpha_1}3^{\alpha_2}37^{\alpha_3}73^n$ is a divisor of $2^{3n}3^{2n}37^n73^n$. Therefore, by Table 1, we have

$$
\beta(G) \leq \left(\frac{\alpha_1}{\text{ord}_{73}(2)} + \frac{\alpha_2}{\text{ord}_{73}(3)} + \frac{\alpha_3}{\text{ord}_{73}(37)}\right) \frac{73}{72} \leq \left(\frac{3n}{9} + \frac{2n}{12} + \frac{n}{9}\right) \frac{73}{72} = \frac{803}{1296}n < n.
$$

In other cases, similarly we get the result. For the details see Table 1.

On the other hand, by the assumptions,

$$
F(G) \cong O_{p_1}(G) \times O_{p_2}(G) \times \cdots \times O_{p_k-1}(G).
$$

If $G$ is solvable, then $C_G(F(G)) \leq F(G)$. The Normalizer-Centralizer Theorem implies that $|G|$ is a divisor of $|F(G)| : |\text{Aut}(F(G))|$. Moreover, using [3], we conclude that $|G|$ is a divisor of $|F(G)| : |\text{GL}(\alpha_1, p_1)| : |\text{GL}(\alpha_2, p_2)| \cdots |\text{GL}(\alpha_{k-1}, p_{k-1})|$.  

12
An upper bound for $\beta(G)$.

| $p$ | $|\text{PSL}(2, p)|$ | An upper bound for $\beta(G)$ |
|-----|------------------|-------------------------------|
| 5   | $2^2 \cdot 3 \cdot 5$ | $\frac{2n}{4} + \frac{n}{4} = \frac{15}{16}$ |
| 17  | $2^4 \cdot 3^2 \cdot 17$ | $\frac{4n}{8} + \frac{2n}{16} = \frac{85}{16}$ |
| 23  | $2^3 \cdot 3 \cdot 11 \cdot 23$ | $\frac{3n}{11} + \frac{n}{22} = \frac{207}{53}$ |
| 37  | $2^2 \cdot 3^2 \cdot 19 \cdot 37$ | $\frac{2n}{36} + \frac{2n}{36} + \frac{n}{36} = \frac{259}{1296}$ |
| 47  | $2^4 \cdot 3 \cdot 23 \cdot 47$ | $\frac{4n}{23} + \frac{n}{23} = \frac{517}{2116}$ |
| 73  | $2^3 \cdot 3^2 \cdot 37 \cdot 73$ | $\frac{3n}{9} + \frac{2n}{12} + \frac{n}{72} = \frac{803}{1296}$ |

Table 1: An upper bound for $\beta(G)$.

Therefore, $p^n$ is a divisor of $|\text{GL}(\alpha_1, p)| \cdot |\text{GL}(\alpha_2, p_2)| \cdots |\text{GL}(\alpha_k, p_k)|$. It also is easy to check for each prime divisor $p_i$ of $p^2 - 1$, we have $\text{ord}_{p^i}(p_i) = p \times \text{ord}_p(p_i)$. Hence, by Lemma 2.2,

$$n = \left\lfloor \frac{\alpha_1}{\text{ord}_p(p_1)} \right\rfloor + \left\lfloor \frac{\alpha_1}{\text{ord}_p(p_1)^2} \right\rfloor + \left\lfloor \frac{\alpha_1}{\text{ord}_p(p_1)^3} \right\rfloor + \cdots$$

$$+ \left\lfloor \frac{\alpha_2}{\text{ord}_p(p_2)} \right\rfloor + \left\lfloor \frac{\alpha_2}{\text{ord}_p(p_2)^2} \right\rfloor + \left\lfloor \frac{\alpha_2}{\text{ord}_p(p_2)^3} \right\rfloor + \cdots$$

$$
\vdots
$$

$$+ \left\lfloor \frac{\alpha_{k-1}}{\text{ord}_p(p_{k-1})} \right\rfloor + \left\lfloor \frac{\alpha_{k-1}}{\text{ord}_p(p_{k-1})} \right\rfloor + \left\lfloor \frac{\alpha_{k-1}}{\text{ord}_p(p_{k-1})} \right\rfloor + \cdots$$

$$< \left( \frac{\alpha_1}{\text{ord}_p(p_1)} + \frac{\alpha_2}{\text{ord}_p(p_2)} + \cdots + \frac{\alpha_{k-1}}{\text{ord}_p(p_{k-1})} \right) \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right)$$

$$= \left( \frac{\alpha_1}{\text{ord}_p(p_1)} + \frac{\alpha_2}{\text{ord}_p(p_2)} + \cdots + \frac{\alpha_{k-1}}{\text{ord}_p(p_{k-1})} \right) \frac{p}{p - 1} < n,$$

which is a contradiction.

Using [12], we have the following result:

**Lemma 3.2.** Let $S$ be a non-abelian simple group with $\pi(S) \subseteq \pi(\text{PSL}(2, p))$, where $p \in \{5, 17, 23, 37, 47, 73\}$. Then $S \cong \text{PSL}(2, p)$.

**Proof. [Proof of Theorem 1.2]**

Lemma 3.1 implies that $G$ is non-solvable. Using Lemma 2.1, $G$ contains a normal series $1 \leq H_1 \leq K_1 \leq G$ such that $K_1/H_1$ is a non-abelian characteristically simple group and $|G/K_1| = |\text{Out}(K_1/H_1)|$.

If $H_1$ is non-solvable, then there exists a normal series $1 \leq H_2 \leq K_2 \leq H_1$ such that $K_2/H_2$ is a non-abelian characteristically simple group and $|H_1/K_2| = |\text{Out}(K_2/H_2)|$. By proceeding, we have the following subnormal series:

$$1 \leq H_m \leq K_m \leq H_{m-1} \leq K_{m-1} \cdots \leq H_2 \leq K_2 \leq H_1 \leq K_1 \leq G = H_0,$$

where $m \geq 1$ is the smallest integer such that $H_m$ is solvable, and so

$$|G| = |H_m| \prod_{i=1}^{m} |K_i/H_i||H_{i-1}/K_i|.$$
Now, we consider the following cases:

(I) Assume that \( p \mid \prod_{i=1}^{m} |H_{i-1}/K_{i}| \). We also know \( p \) does not divide \( |\text{Out}(\text{PSL}(2, p))| \). Therefore, there exists \( 1 \leq i \leq m \) such that

\[
p \mid |H_{i-1}/K_{i}| = |\text{Out}(\text{PSL}(2, p))|^{n_{i}}! \implies p \mid n_{i}!,
\]

and so \( p \leq n_{i} \). Since \( n_{i} \leq n \leq p \), \( n \) is equal to \( p \). Thus, \( p^{n} \mid |K_{i}/H_{i}| \). As a result, \( p^{n+1} \) divides \( |G| \), a contradiction. Hence, \( p \mid \prod_{i=1}^{m} |H_{i-1}/K_{i}| \).

(II) If \( p \mid |H_{m}| \), then there exists a natural number \( a \) such that \( p^{a} \mid |H_{m}| \). Therefore, \( p^{a} \) is a divisor of

\[
t = |H_{m}| \prod_{i=1}^{m} |H_{i-1}/K_{i}| = |G|/\prod_{i=1}^{m} |K_{i}/H_{i}|.
\]

Hence, \( t \) is a divisor of \( |\text{PSL}(2, p)|^{n}/|\text{PSL}(2, p)|^{c} \), where \( p^{c} \mid \prod_{i=1}^{m} |K_{i}/H_{i}| \), and \( c \) is a natural number. Hence, \( n = a + c \), and so \( |H_{m}| \) is a divisor of \( |\text{PSL}(2, p)|^{a} \).

By Lemma 3.1, we get a contradiction. Thus, \( p \nmid |H_{m}| \).

By the above discussion, \( p^{a} \mid \prod_{i=1}^{m} |K_{i}/H_{i}| \), and since each \( K_{i}/H_{i} \) is a direct product of \( n_{i} \) copies of \( \text{PSL}(2, p) \), we get \( H_{m-1} = K_{i} \), where \( 1 \leq i \leq m \). Then, \( \sum_{i=1}^{m} n_{i} = n \). Thus, we conclude that

\[
1 = H_{m-1} \leq H_{m-2} \leq \cdots \leq H_{2} \leq H_{1} \leq G \leq H_{0}.
\]

Using Lemma 2.3, since \( H_{m-1} \cong H_{m} \cong \text{PSL}(2, p)^{n_{m}} \) and \( H_{m-2}/H_{m-1} \cong H_{m-1}/H_{m} \) we get \( H_{m-2} \cong H_{m-1} \), and so \( G \) is isomorphic to \( \text{PSL}(2, p)^{n} \).

By [2, Corollary 11.29], we deduce that if \( a \in \text{cd}(G) \) such that \( a_{p} = |G|_{p} \), then \( O_{p}(G) = 1 \). Therefore, we have the following corollary:

**Corollary 3.3.** Let \( G \) be a finite group, \( p \in \{5, 17, 23, 37, 47, 73\} \) and \( n \leq p \), where \( n \) is a natural number. Then the following are equivalent.

1. \( G \) is isomorphic to \( \text{PSL}(2, p)^{n} \);
2. \( |G| = |\text{PSL}(2, p)^{n}| \) and \( p^{n} \in V(\Gamma(G)) \);
3. \( |G| = |\text{PSL}(2, p)^{n}| \) and \( p^{n} \in \text{cd}(G) \);
4. \( |G| = |\text{PSL}(2, p)^{n}| \) and \( \Gamma(G) = \Gamma(\text{PSL}(2, p)^{n}) \).

**Acknowledgement**

The authors are very thankful to the referee for valuable comments.

**References**


