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Original Article

Recognition by degree prime-power graph and order of some characteristically simple groups

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ABSTRACT: In this paper, by the order of a group and triviality of $O_p(G)$ for some prime p, we give a new characterization for some characteristically simple groups. In fact, we prove that if $p \in \{5, 17, 23, 37, 47, 73\}$ and $n \leqslant p$, where n is a natural number, then $G \cong \mathrm{PSL}(2, p)^n$ if and only if $|G| = |\mathrm{PSL}(2, p)|^n$ and $O_p(G) = 1$.

Recently in [Qin, Yan, Shum and Chen, Comm. Algebra, 2019], the degree prime-power graph of a finite group have been introduced and it is proved that the Mathieu groups are uniquely determined by their degree prime-power graphs and orders. As a consequence of our results, we show that $\mathrm{PSL}(2, p)^n$, where $p \in \{5, 17, 23, 37, 47, 73\}$ and $n \leqslant p$ are uniquely determined by their degree prime-power graphs and orders.

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1. Introduction

Throughout the paper, G is a finite group. The set of all prime divisors of $|G|$ is denoted by $\pi(G)$. By G^n, we mean the direct product of n copies of G. Let $\mathrm{cd}(G)$ be the set of irreducible character degrees of G, and $\rho(G)$ the set of primes dividing the elements in $\mathrm{cd}(G)$. Some graphs are defined concerning to the irreducible characters of a finite group. We refer the readers to a survey by Lewis [7] for results concerning graphs associated with character degrees. The character degree graph of G, which is denoted by $\Delta(G)$ was introduced by Manz *et al.* in 1998 (see [8]). The vertex set of $\Delta(G)$ is $\rho(G)$ and there exists an edge between two distinct elements $a, b \in \rho(G)$, if ab divides some irreducible character degree in $\mathrm{cd}(G)$.

In [4, 5, 6], the authors proved that $A_5, A_5 \times A_5$ and some other groups are characterizable by their character degree graphs and orders. Obviously, A_5^n, where $n > 2$, is not uniquely determined by $\Delta(A_5^n)$ and $|A_5^n|$, since A_5^n and $A_5 \times A_5 \times L$, where L is a group of order 60^{n-2} have the same order and character degree graph (see Figure 1). There also exist some simple groups, say M_{12}, which are not characterizable by their character degree graphs and orders. For this reason, Qin *et al.* in [10] have introduced a new graph related to irreducible characters of a finite group as follows and they showed that the Mathieu groups are characterizable by this graph and order.

Notation. Let m and a be integers such that $(a, m) = 1$. Then, $\mathrm{ord}_m(a)$ denotes the smallest positive integer e such that $a^e \equiv 1 \pmod m$. In addition, for a prime p, we write $p^k \| n$, whenever $p^k \mid n$ but $p^{k+1} \nmid n$. In this case, we also write $n_p = p^k$.

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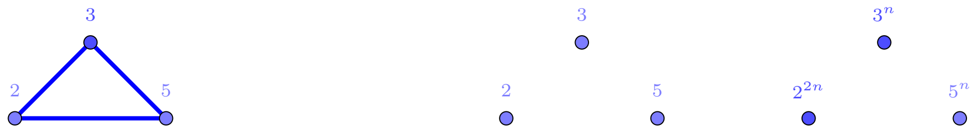
Definition 1.1. *The degree prime-power graph $\Gamma(G)$ is defined as follow:*
For each $p \in \rho(G)$ let $b(p) = max\{a_p \,|a \in \mathrm{cd}(G)\}$. The vertex set of $\Gamma(G)$ is $V = \{b(p)\,|p \in \rho(G)\}$ and there is an edge between distinct vertices $x, y \in V$ if xy divides an element of $\mathrm{cd}(G)$.



$\Delta(A_5^n) = \Delta(A_5^n \times L); n \geq 2, \pi(L) \subseteq \{2,3,5\}$ $\Delta(A_5)$ $\Gamma(A_5^n)$

Figure 1: Graphs.

In this paper, for a characteristically simple group G we present a new characterization based on the order of G and $O_p(G)$, for some $p \in \pi(G)$. Our main result is:

Theorem 1.2. *Let G be a finite group, and $M = \mathrm{PSL}(2,p)^n$, where $p \in \{5, 17, 23, 37, 47, 73\}$, $n \leq p$. Then $G \cong M$ if and only if $|G| = |M|$ and $|O_p(G)| = |O_p(M)| = 1$.*

As a consequence of our results, we show that these groups are uniquely determined by their degree prime-power graphs and orders.

2. Preliminary Results

Lemma 2.1. [11, Lemma] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 2.2. [9, Theorems 3.6] *Let p be an odd prime, and let $a \neq \pm 1$ be an integer not divisible by p. Let also d be the order of a modulo p and k_0 the largest integer such that $a^d \equiv 1 \pmod{p^{k_0}}$. Then the order of a modulo p^k is d for $k = 1, ..., k_0$ and dp^{k-k_0} for $k > k_0$.*

Lemma 2.3. *[1, Lemma 2.2] Let S be a finite non-abelian simple group, and p_0 be the largest prime divisor of $|S|$. If G is an extension of S^m by S^n, where $m + n \leqslant p_0$, then $G \cong S^{m+n}$.*

Lemma 2.4. *Let S be a subnormal subgroup of G, and $O_p(G) = 1$, where $p \in \pi(G)$. Then $O_p(S) = 1$.*

Proof. It is straightforward. □

3. Main Results

Lemma 3.1. *Let $p \in \{5, 17, 23, 37, 47, 73\}$, $|G|$ be a divisor of $|\mathrm{PSL}(2,p)|^n$ and $p^n \mid |G|$, where n is a natural number. If $O_p(G) = 1$, then G is non-solvable.*

Proof. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}} p^n$ and define

$$\beta(G) = \left(\frac{\alpha_1}{\mathrm{ord}_p(p_1)} + \frac{\alpha_2}{\mathrm{ord}_p(p_2)} + \cdots + \frac{\alpha_{k-1}}{\mathrm{ord}_p(p_{k-1})}\right)\frac{p}{p-1}.$$

Firstly, in each case we prove that $\beta(G) < n$. Assume that $p = 73$. Thus, $|G| = 2^{\alpha_1} 3^{\alpha_2} 37^{\alpha_3} 73^n$ is a divisor of $2^{3n} 3^{2n} 37^n 73^n$. Therefore, by Table 1, we have

$$\begin{aligned}\beta(G) &= \left(\frac{\alpha_1}{\mathrm{ord}_{73}(2)} + \frac{\alpha_2}{\mathrm{ord}_{73}(3)} + \frac{\alpha_3}{\mathrm{ord}_{73}(37)}\right)\frac{73}{72} \\ &\leqslant \left(\frac{3n}{9} + \frac{2n}{12} + \frac{n}{9}\right)\frac{73}{72} = \frac{803}{1296}n < n.\end{aligned}$$

In other cases, similarly we get the result. For the details see Table 1.
On the other hand, by the assumptions,

$$\mathrm{F}(G) \cong O_{p_1}(G) \times O_{p_2}(G) \times \cdots \times O_{p_{k-1}}(G).$$

If G is solvable, then $C_G(\mathrm{F}(G)) \leq \mathrm{F}(G)$. The Normalizer-Centralizer Theorem implies that $|G|$ is a divisor of $|\mathrm{F}(G)| \cdot |\mathrm{Aut}(\mathrm{F}(G))|$. Moreover, using [2], we conclude that $|G|$ is a divisor of $|\mathrm{F}(G)| \cdot |\mathrm{GL}(\alpha_1, p_1)| \cdot |\mathrm{GL}(\alpha_2, p_2)| \cdots |\mathrm{GL}(\alpha_{k-1}, p_{k-1})|$.

p	$\vert\mathrm{PSL}(2,\mathrm{p})\vert$	An upper bound for $\beta(G)$
5	$2^2\cdot 3\cdot 5$	$(\frac{2n}{4}+\frac{n}{4})\frac{5}{4}=\frac{15}{16}n$
17	$2^4\cdot 3^2\cdot 17$	$(\frac{4n}{8}+\frac{2n}{16})\frac{17}{16}=\frac{85}{128}n$
23	$2^3\cdot 3\cdot 11\cdot 23$	$(\frac{3n}{11}+\frac{n}{11}+\frac{n}{22})\frac{23}{22}=\frac{207}{484}n$
37	$2^2\cdot 3^2\cdot 19\cdot 37$	$(\frac{2n}{36}+\frac{2n}{18}+\frac{n}{36})\frac{37}{36}=\frac{259}{1296}n$
47	$2^4\cdot 3\cdot 23\cdot 47$	$(\frac{4n}{23}+\frac{n}{23}+\frac{n}{46})\frac{47}{46}=\frac{517}{2116}n$
73	$2^3\cdot 3^2\cdot 37\cdot 73$	$(\frac{3n}{9}+\frac{2n}{12}+\frac{n}{9})\frac{73}{72}=\frac{803}{1296}n$

Table 1: An upper bound for $\beta(G)$.

Therefore, p^n is a divisor of $|\mathrm{GL}(\alpha_1,p_1)|\cdot|\mathrm{GL}(\alpha_2,p_2)|\cdots|\mathrm{GL}(\alpha_{k-1},p_{k-1})|$. It also is easy to check for each prime divisor p_i of p^2-1, we have $\mathrm{ord}_{p^2}(p_i)=p\times\mathrm{ord}_p(p_i)$. Hence, by Lemma 2.2,

$$\begin{aligned}
n = &\lfloor\frac{\alpha_1}{\mathrm{ord}_p(p_1)}\rfloor+\lfloor\frac{\alpha_1}{\mathrm{ord}_{p^2}(p_1)}\rfloor+\lfloor\frac{\alpha_1}{\mathrm{ord}_{p^3}(p_1)}\rfloor+\cdots\\
&+\lfloor\frac{\alpha_2}{\mathrm{ord}_p(p_2)}\rfloor+\lfloor\frac{\alpha_2}{\mathrm{ord}_{p^2}(p_2)}\rfloor+\lfloor\frac{\alpha_2}{\mathrm{ord}_{p^3}(p_2)}\rfloor+\cdots\\
&\vdots\\
&+\lfloor\frac{\alpha_{k-1}}{\mathrm{ord}_p(p_{k-1})}\rfloor+\lfloor\frac{\alpha_{k-1}}{\mathrm{ord}_{p^2}(p_{k-1})}\rfloor+\lfloor\frac{\alpha_{k-1}}{\mathrm{ord}_{p^3}(p_{k-1})}\rfloor+\cdots\\
&<\left(\frac{\alpha_1}{\mathrm{ord}_p(p_1)}+\frac{\alpha_2}{\mathrm{ord}_p(p_2)}+\cdots+\frac{\alpha_{k-1}}{\mathrm{ord}_p(p_{k-1})}\right)(1+\frac{1}{p}+\frac{1}{p^2}+\cdots)\\
&=\left(\frac{\alpha_1}{\mathrm{ord}_p(p_1)}+\frac{\alpha_2}{\mathrm{ord}_p(p_2)}+\cdots+\frac{\alpha_{k-1}}{\mathrm{ord}_p(p_{k-1})}\right)\frac{p}{p-1}<n,
\end{aligned}$$

which is a contradiction. □

Using [12], we have the following result:

Lemma 3.2. *Let S be a non-abelian simple group with $\pi(S)\subseteq\pi(\mathrm{PSL}(2,p))$, where $p\in\{5,17,23,37,47,73\}$. Then $S\cong\mathrm{PSL}(2,p)$.*

Proof of Theorem 1.2. Lemma 3.1 implies that G is non-solvable. Using Lemma 2.1, G contains a normal series $1\trianglelefteq H_1\trianglelefteq K_1\trianglelefteq G$ such that K_1/H_1 is a non-abelian characteristically simple group and $|G/K_1|\mid|\mathrm{Out}(K_1/H_1)|$.

If H_1 is non-solvable, then there exists a normal series $1\trianglelefteq H_2\trianglelefteq K_2\trianglelefteq H_1$ such that K_2/H_2 is a non-abelian characteristically simple group and $|H_1/K_2|\mid|\mathrm{Out}(K_2/H_2)|$. By proceeding, we have the following subnormal series:

$$1\trianglelefteq H_m\trianglelefteq K_m\trianglelefteq H_{m-1}\trianglelefteq K_{m-1}\cdots\trianglelefteq H_2\trianglelefteq K_2\trianglelefteq H_1\trianglelefteq K_1\trianglelefteq G=H_0, \quad (1)$$

where $m\geqslant 1$ is the smallest integer such that H_m is solvable, and so

$$|G|=|H_m|\prod_{i=1}^{m}|K_i/H_i||H_{i-1}/K_i|.$$

We note K_i/H_i is a direct product of n_i copies of a non-abelian simple group S_i such that $|H_{i-1}/K_i|\mid|\mathrm{Out}(K_i/H_i)|$. Lemma 3.2 leads us to $S_i\cong\mathrm{PSL}(2,p)$.

Now, we consider the following cases:

(I) Assume that $p\mid\prod_{i=1}^{m}|H_{i-1}/K_i|$. We also know p does not divide $|\mathrm{Out}(\mathrm{PSL}(2,p))|$. Therefore, there exists $1\leqslant i\leqslant m$ such that

$$p\mid|H_{i-1}/K_i|=|\mathrm{Out}(\mathrm{PSL}(2,p))|^{n_i}n_i!\Longrightarrow p\mid n_i!,$$

and so $p \leqslant n_i$. Since $n_i \leqslant n \leqslant p$, n is equal to p. Thus, $p^n \mid |K_i/H_i|$. As a result, p^{n+1} divides $|G|$, a contradiction. Hence, $p \nmid \prod_{i=1}^m |H_{i-1}/K_i|$.

(II) If $p \mid |H_m|$, then there exists a natural number a such that $p^a \| |H_m|$. Therefore, p^a is a divisor of

$$t = |H_m| \prod_{i=1}^{m} |H_{i-1}/K_i| = |G| / \prod_{i=1}^{m} |K_i/H_i|.$$

Hence, t is a divisor of $|\mathrm{PSL}(2,p)|^n/|\mathrm{PSL}(2,p)|^c$, where $p^c \parallel \prod_{i=1}^m |K_i/H_i|$, and c is a natural number. Hence, $n = a + c$, and so $|H_m|$ is a divisor of $|\mathrm{PSL}(2,p)|^a$.

Lemma 2.4 implies that $O_p(H_m) = 1$. By Lemma 3.1, we get a contradiction. Thus, $p \nmid |H_m|$.

By the above discussion, $p^n \parallel \prod_{i=1}^m |K_i/H_i|$, and since each K_i/H_i is a direct product of n_i copies of $\mathrm{PSL}(2,p)$, we get $H_m = 1$, $H_{i-1} = K_i$, where $1 \leqslant i \leqslant m$. Then, $\sum_{i=1}^m n_i = n$. Thus, We conclude that

$$1 = H_m \trianglelefteq H_{m-1} \trianglelefteq H_{m-2} \cdots \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq G = H_0.$$

Using Lemma 2.3, since $H_{m-1} \cong \mathrm{PSL}(2,p)^{n_m}$ and $H_{m-2}/H_{m-1} \cong \mathrm{PSL}(2,p)^{n_{m-1}}$ we get $H_{m-2} \cong \mathrm{PSL}(2,p)^{n_m+n_{m-1}}$, and so G is isomorphic to $\mathrm{PSL}(2,p)^n$. $\square$

By [3, Corollary 11.29], we deduce that if $a \in \mathrm{cd}(G)$ such that $a_p = |G|_p$, then $O_p(G) = 1$. Therefore, we have the following corollary:

Corollary 3.3. *Let G be a finite group, $p \in \{5, 17, 23, 37, 47, 73\}$ and $n \leqslant p$, where n is a natural number. Then the following are equivalent.*

(1) G is isomorphic to $\mathrm{PSL}(2,p)^n$;

(2) $|G| = |\mathrm{PSL}(2,p)^n|$ and $p^n \in V(\Gamma(G))$;

(3) $|G| = |\mathrm{PSL}(2,p)^n|$ and $p^n \in \mathrm{cd}(G)$;

(4) $|G| = |\mathrm{PSL}(2,p)^n|$ and $\Gamma(G) = \Gamma(\mathrm{PSL}(2,p)^n)$.

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