



On the rank of the holomorphic solutions of PDE associated to directed graphs

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ABSTRACT: Let G be a directed graph with m vertices and n edges, $I(\mathbf{B})$ the binomial ideal associated to the incidence matrix \mathbf{B} of the graph G , and I_L the lattice ideal associated to the columns of the matrix \mathbf{B} . Also let \mathbf{B}_i be a submatrix of \mathbf{B} after removing the i th column. In this paper it is determined that which minimal prime ideals of $I(\mathbf{B}_i)$ are Andean or toral. Then we study the rank of the space of solutions of binomial D -module associated to $I(\mathbf{B}_i)$ as \mathbf{A} -graded ideal, where \mathbf{A} is a matrix that, $\mathbf{A}\mathbf{B}_i = 0$. Afterwards, we define a miniaml cellular cycle and prove that for computing this rank it is enough to consider these components of G . We introduce some bounds for the number of the vertices of the convex hull generated by the columns of the matrix \mathbf{A} . Finally an algorithm is introduced by which we can compute the volume of the convex hull corresponded to a cycles with k diagonals, so by Theorem 2.1 the rank of $\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}$ can be computed.

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1. Introduction

The main object of study in this article is the binomial D -module, introduced in [4]. In the late 1980s, Gelfand, Graev and Zelevinsky introduced a class of systems of linear partial differential equations closely related to toric varieties [8]. These systems, called GKZ systems, or \mathbf{A} -hypergeometric systems, are constructed from a $d \times n$ integer matrix \mathbf{A} of rank d and a complex parameter vector $\beta \in \mathbb{C}^d$, and are denoted by $H_{\mathbf{A}}(\beta)$.

Convention 1.1. Throughout the paper the matrices \mathbf{A} and \mathbf{B} have the following properties:

1. $\mathbf{A} = (a_{ij}) \in \mathbb{Z}^{d \times n}$ denotes an integer $d \times n$ matrix of rank d whose columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ all lie in a single open linear half-space of \mathbb{R}^d ; equivalently, the cone generated by the columns of \mathbf{A} is pointed (contains no lines), and all of the \mathbf{a}_i 's are nonzero. We also assume that $\mathbb{Z}\mathbf{A} = \mathbb{Z}^d$; that is, the columns of \mathbf{A} span \mathbb{Z}^d as a lattice.
2. Let $\mathbf{B} = (b_{jk}) \in \mathbb{Z}^{n \times m}$ be an integer matrix of full rank $m \leq n$. Assume that every nonzero element of the column-span of \mathbf{B} over the integers \mathbb{Z} is mixed, meaning that it has at least one positive and one negative entry; in particular, the columns of \mathbf{B} are mixed. We write $\mathbf{b}_1, \dots, \mathbf{b}_n$ for the rows of \mathbf{B} . Having chosen \mathbf{B} , we set $d = n - m$ and pick a matrix $\mathbf{A} \in \mathbb{Z}^{d \times n}$ whose columns span \mathbb{Z}^d as a lattice, such that $\mathbf{A}\mathbf{B} = 0$.

Definition 1.2. Let $\mathbf{A} \in \mathbb{Z}^{d \times n}$, so $\mathbb{Z}\mathbf{A} \subseteq \mathbb{Z}^d$ is a subgroup. A ring R is \mathbf{A} -graded if R is a direct sum of homogeneous components

$$R = \bigoplus_{\alpha \in \mathbb{Z}\mathbf{A}} R_{\alpha}; \quad R_{\alpha}R_{\beta} \subseteq R_{\alpha+\beta}.$$

An ideal in an \mathbf{A} -graded ring is \mathbf{A} -graded if it is generated by homogeneous elements.

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Definition 1.3 (Definition 1.3 [4]). For each $i \in \{1, \dots, d\}$, the i th Euler operator is;

$$E_i = a_{i1}x_1\partial_1 + \dots + a_{in}x_n\partial_n,$$

where ∂_i is $\frac{\partial}{\partial x_i}$. Given a vector $\beta \in \mathbb{C}^d$, we write $\mathbf{E} - \beta$ for the sequence $E_1 - \beta_1, \dots, E_d - \beta_d$. For an \mathbf{A} -graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_{\mathbf{A}}(I, \beta)$, the left ideal $I + \langle \mathbf{E} - \beta \rangle$ in the Weyl algebra D . The binomial D -module associated to I is $\frac{D}{H_{\mathbf{A}}(I, \beta)}$.

Given \mathbf{A} as in Convention 1.1, these are the left D -ideals $H_{\mathbf{A}}(I_{\mathbf{A}}, \beta)$, also denoted by $H_{\mathbf{A}}(\beta)$, where

$$I_{\mathbf{A}} = \langle \partial^{\mathbf{u}} - \partial^{\mathbf{v}} : \mathbf{A} \cdot \mathbf{u} = \mathbf{A} \cdot \mathbf{v} \rangle \subseteq \mathbb{C}[\partial_1, \dots, \partial_n].$$

The \mathbf{A} -hypergeometric systems have many applications; for example, they arise naturally in the moduli theory of Calabi-Yau complete intersections in toric varieties, and (therefore) they play an important role in applications of mirror symmetry in mathematical physics [4].

Definition 1.4 (Definition 1.8 [4]). Fix integer matrices \mathbf{B} and \mathbf{A} as in Convention 1.1, and let $I(\mathbf{B})$ be the lattice basis ideal corresponding to this matrix, that is, the ideal in $\mathbb{C}[\partial]$ generated by the binomials

$$\prod_{b_{jk} > 0} \partial_{x_j}^{b_{jk}} - \prod_{b_{jk} < 0} \partial_{x_j}^{-b_{jk}}, \quad \text{for } 1 \leq k \leq m.$$

The binomial Horn system with parameter β is the left ideal $H(\mathbf{B}, \beta) = H_{\mathbf{A}}(I(\mathbf{B}), \beta)$ in the Weyl algebra $D = D_n$.

If $L \subseteq \mathbb{Z}^n$ is a sublattice, then the lattice ideal of L is

$$I_L = \langle \partial^{\mathbf{u}^+} - \partial^{\mathbf{u}^-} : \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in L \rangle.$$

Here and henceforth, \mathbf{u}^+ has i th coordinate u_i if $u_i \geq 0$ and 0 otherwise. The vector $\mathbf{u}^- \in \mathbb{N}^n$ is defined by $\mathbf{u}^+ - \mathbf{u}^- = \mathbf{u}$, or, equivalently, $\mathbf{u}^- = (-\mathbf{u})^+$. More general than I_L are the ideals

$$I_{\rho} = \langle \partial^{\mathbf{u}^+} - \partial^{\mathbf{u}^-} : \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in L \rangle$$

for any partial character $\rho : L \rightarrow \mathbb{C}^*$ of \mathbb{Z}^n , which includes the data of both its domain lattice $L \subseteq \mathbb{Z}^n$ and the map to \mathbb{C}^* . The ideal I_{ρ} is prime if and only if L is a saturated sublattice of \mathbb{Z}^n , meaning that L equals its saturation

$$\text{sat}(L) = (\mathbb{Q}L) \cap \mathbb{Z}^n,$$

where $\mathbb{Q}L = \mathbb{Q} \otimes_{\mathbb{Z}} L$ is the rational vector space spanned by L in \mathbb{Q}^n . Every binomial prime ideal in $\mathbb{C}[\partial]$ has the form

$$I_{\rho, J} = I_{\rho} + \langle \partial_j : j \notin J \rangle$$

for some saturated partial character ρ (i.e., whose domain is a saturated sublattice) and subset $J \subseteq \{1, \dots, n\}$ such that the binomial generators of I_{ρ} only involve variables $\partial_j, j \in J$.

Lemma 1.5. (Lemma 3.4 [4]) Fix a partial character $\rho : L \rightarrow \mathbb{C}^*$ for a saturated sublattice $L \subseteq \mathbb{Z}^J \subseteq \mathbb{Z}^n$. Let $C_{\rho, J}$ be an \mathbf{A} -graded binomial $I_{\rho, J}$ -primary ideal. Then $L \subseteq \mathbb{Z}_J \cap \ker_{\mathbb{Z}}(\mathbf{A}) = \ker_{\mathbb{Z}}(\mathbf{A}_J)$, the Krull dimension satisfies $\dim(\mathbb{C}[\partial]/I_{\rho, J}) \geq \text{rank}(\mathbf{A}_J)$, and the following are equivalent.

- The Hilbert function $\mathbb{Z}\mathbf{A} \rightarrow \mathbb{N}$ defined by $\alpha \rightarrow \dim_{\mathbb{C}}(\mathbb{C}[\partial]/C_{\rho, J})_{\alpha}$ is bounded above.
- The homomorphism $\mathbb{Z}^J/L \rightarrow \mathbb{Z}\mathbf{A}_J \subseteq \mathbb{Z}^d$ is injective.
- $L = \ker_{\mathbb{Z}}(\mathbf{A}_J)$.
- $\dim(\mathbb{C}[\partial]/I_{\rho, J}) = \text{rank}(\mathbf{A}_J)$.

When these conditions are satisfied, the module $\mathbb{C}[\partial]/C_{\rho, J}$ and the lattice L are called toral, the ideal $I_{\rho, J}$ is called a toral prime, and $C_{\rho, J}$ is called a toral (primary) component. When these conditions are not satisfied, substitute Andean for “toral” above.

In 2010 Dickenstein, Matusevich, and Miller presented and proved the following theorem [4]:

Theorem 1.6. (Theorem 6.10 [4]) If $\mathcal{Z}_{\text{Andean}}(I) \neq \mathbb{C}^d$, then $H_{\mathbf{A}}(I, \beta)$ has minimal rank at β if and only if $-\beta$ lies outside of the jump arrangement $\mathcal{Z}_{\text{jump}}(I)$, and this minimal rank is

$$\text{rank}\left(\frac{D}{H_{\mathbf{A}}(I, \beta)}\right) = \sum_{I_{\rho, J} \text{ total of dim } d} \mu_{\rho, J} \cdot \text{vol } \mathbf{A}_J.$$

Where $\mu_{\rho, J}$ be multiplicity of $I_{\rho, J}$ in I (or equivalently, in the primary component $C_{\rho, J}$ of I) and $\text{vol}(\mathbf{A}_J)$ the volume of the convex hull of \mathbf{A}_J and the origin, normalized so that a lattice simplex in the group $\mathbb{Z}\mathbf{A}_J$ generated by the columns of \mathbf{A}_J has volume 1.

In order to obtain the necessary preliminaries for stating and proving the Proposition 2.8 and Theorem 2.1, we review some concepts as follow.

Let P be a lattice polytope of dimension d , i.e. a convex polytope in \mathbb{R}^d whose vertices are elements of \mathbb{Z}^d and whose affine span has dimension d , and P° denote the interior of P . Given a positive integer n , the numerical functions $i(P, n)$ and $\bar{i}(P, n)$ are defined as follow:

$$i(P, n) = |nP \cap \mathbb{Z}^d|, \quad \bar{i}(P, n) = |n(P^\circ) \cap \mathbb{Z}^d|.$$

Here $nP = \{n\alpha : \alpha \in P\}$ and $|X|$ is the cardinality of a finite set X . Ehrhart in [7] stated the following properties:

1. $i(P, n)$ is a polynomial in n of degree d (and thus in particular $i(P, n)$ can be defined for every integer n);
2. $i(P, 0) = 1$;
3. $\bar{i}(P, n) = (-1)^d i(P, -n)$ for every integer $n \geq 0$.

Let

$$\text{Ehr}P(x) = \sum_{n \geq 0} i(P, n)x^n = \frac{\sum_{j=0}^d h_j^* x^j}{(1-x)^{d+1}},$$

denote the rational generating function for this polynomial, called the Ehrhart series of P .

Definition 1.7. For two polytopes $P \subseteq \mathbb{R}^{d_P}$ and $Q \subseteq \mathbb{R}^{d_Q}$ of dimension d_P and d_Q , define the free sum to be

$$P \oplus Q = \text{conv}\{(0_P \times Q) \cup (P \times 0_Q)\} \subseteq \mathbb{R}^{d_P+d_Q}.$$

Definition 1.8. Let P be a lattice polytope in \mathbb{R}^{d_P} . The following set is called dual of P ;

$$P^\Delta = \{\mathbf{x} \in \mathbb{R}^{d_P} : \mathbf{x} \cdot \mathbf{p} \leq 1 \text{ for all } \mathbf{p} \in P\}.$$

A lattice polytope whose dual is lattice polytope, called reflexive polytope.

Batyrev and Hibi in [1] and [10] respectively, proved the following lemma;

Lemma 1.9. P is reflexive if and only if P is a lattice polytope with $0 \in P^\circ$ that satisfies one of the following (equivalent) conditions:

1. P^Δ is a lattice polytope.
2. $\bar{i}(P, n+1) = i(P, n)$ for all $n \in \mathbb{N}$, i.e. all lattice points in \mathbb{R}^{d_P} sit on the boundary of some non-negative integral dilate of P .
3. $h_i^* = h_{d_P-i}^*$ for all i , where h_i^* is the i th coefficient in the numerator of the Ehrhart series for P .

Proposition 1.10. (Corollary 3.6. [9]) Let P be a d -dimensional reflexive polytope. Then

$$\text{vol}(P) = \frac{1}{d!} \sum_{b=0}^c (-1)^{b+c} \left(\binom{d}{c-b} + (-1)^{d-1} \binom{d}{c+b+1} \right) i(P, b),$$

where $c := \lfloor \frac{d}{2} \rfloor$.

Braun in [2] proved the following theorem;

Theorem 1.11. If P is a d_P -dimensional reflexive polytope in \mathbb{R}^{d_P} and Q is a d_Q -dimensional lattice polytope in \mathbb{R}^{d_Q} with $0 \in Q^\circ$, then

$$\text{Ehr}((P \oplus Q)(x)) = (1-x)\text{Ehr}P(x)\text{Ehr}Q(x).$$

Definition 1.12. Let P be a lattice polytope. A vertex of P is called primitive, if no lattice point lies strictly between the origin and the vertex.

Convention 1.13. The polytope corresponded to the matrix \mathbf{A} , is denoted by $P(\mathbf{A})$.

Proposition 1.14. Let P be a lattice polytope with 0 in its interior. P is reflexive if and only if each vertex is a primitive lattice point.

Proof. It is straightforward corollary of Lemma 1.9. □

In this paper we consider a directed graph G whose vertices have both input and output edges, and let \mathbf{B} be its incidence matrix, that is

$$b_{ij} = \begin{cases} -1 & e_j \text{ exits from } v_i \\ 1 & e_j \text{ inters to } v_i \\ 0 & \text{otherwise} \end{cases}$$

and $I(\mathbf{B})$ is defined as follow;

$$I(\mathbf{B}) = \langle \partial^{\mathbf{u}_i^+} - \partial^{\mathbf{u}_i^-} \mid \mathbf{u}_i = \mathbf{u}_i^+ - \mathbf{u}_i^-, 1 \leq i \leq m, \mathbf{u}_i' \text{ s are the columns of } \mathbf{B} \rangle.$$

Suppose that \mathbf{B}_i is a submatrix of \mathbf{B} after removing the i th column. Assume that L is a lattice which generated by the column vectors of the matrix \mathbf{B} . It is shown that I_L is a toral minimal prime of $I(\mathbf{B}_i)$ and the others are Andean. Afterwards, we define a minimal cellular cycle and prove that for calculating the rank of $H(\mathbf{B}_i, \beta)$, it is enough to consider the minimal cellular cycle components of graph G . We introduce some bounds for the number of the vertices of the convex hull generated by the columns of the matrix \mathbf{A} . Finally an algorithm is introduced by which we can compute the volume of the convex hull corresponded to a cycles with k diagonals, so by Theorem 2.1 the rank of $\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}$ can be computed.

2. Main Results

By noting to Theorem1.6 and using Lemma2.4 and Lemma2.5, for generic parameters, we have:

$$\text{rank}\left(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}\right) = \text{vol}(\mathbf{A}).$$

Hence by the following Theorem, for computing $\text{rank}\left(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}\right)$, we consider the minimal cellular cycles, G_1, \dots, G_t , calculate the volumes of corresponded convex hulls, and finally multiply all of them to compute the volume of \mathbf{A} .

Theorem 2.1. Let G be a directed graph with m vertices and n edges. Suppose that G_1, \dots, G_t are the minimal cellular cycles of $G \setminus \{v_i\}$. Also let $\mathbf{A}_{G_1}, \dots, \mathbf{A}_{G_t}$ be, respectively, the matrices corresponded to the minimal cellular cycles. For generic parameters, we have:

$$\text{rank}\left(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}\right) = \prod_{i=1}^t \text{vol}(\mathbf{A}_{G_i}).$$

In the following, we will mention some necessary facts to prove this Theorem.

Lemma 2.2. The lattice L is saturated.

Proof. Since for all $i \in \{1, \dots, m\}$,

$$-\mathbf{u}_i = \mathbf{u}_1 + \dots + \mathbf{u}_{i-1} + \mathbf{u}_{i+1} + \dots + \mathbf{u}_m,$$

without loss of generality, set $L = \langle \mathbf{u}_1, \dots, \mathbf{u}_{m-1} \rangle$. Let $b \in \mathbb{Z}$ and $\mathbf{Y} = (y_1, \dots, y_n) \in \mathbb{Z}^n$ such that $b\mathbf{Y} \in L$, that is:

$$\exists c_1, \dots, c_{m-1} \in \mathbb{Z}; b\mathbf{Y} = c_1\mathbf{u}_1 + \dots + c_{m-1}\mathbf{u}_{m-1}.$$

Without loss of generality suppose in the first column of \mathbf{B} , the first entry is 1 and the last one is -1, then $by_1 = c_1$. Again without loss of generality suppose in the second column, the first entry is -1 and the second one is 1, we have:

$$by_2 = c_2 - c_1 = c_2 - by_1 \implies c_2 = by_2.$$

Continuing this way, conclude that all of c_i 's are multiplied by b , so

$$\mathbf{Y} = \frac{c_1}{b}\mathbf{u}_1 + \dots + \frac{c_{m-1}}{b}\mathbf{u}_{m-1} \in L.$$

Then L is a saturated lattice.

□

Lemma 2.3. *There exists a $d \times n$ matrix \mathbf{A} of rank d such that for all $i, 1 \leq i \leq m, \mathbf{A}\mathbf{B}_i = 0$.*

Proof. Since the lattice generated by the columns of \mathbf{B} is the same as the lattices generated by the columns of each $\mathbf{B}_i, 1 \leq i \leq m$, without loss of generality, we put:

$$L = \langle \mathbf{u}_1, \dots, \mathbf{u}_{m-1} \rangle.$$

By Lemma 2.2, L is a saturated lattice, so I_L is a prime binomial ideal, hence there is some $d \times n$ matrix \mathbf{A} that, $L = \ker_{\mathbb{Z}} \mathbf{A}$. So for all $i, 1 \leq i \leq m, \mathbf{A}\mathbf{B}_i = 0$. Now because \mathbf{B}_i is full rank and $d = n - m + 1$, the matrix \mathbf{A} is full rank too.

□

The matrix \mathbf{A} induces a \mathbb{Z}^d -grading of the polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n] = \mathbb{C}[\partial]$, called a \mathbf{A} -grading, by setting $\deg(\partial_i) = a_i$, where a_i 's are the columns of the matrix \mathbf{A} . Let I be a binomial ideal of $\mathbb{C}[\partial]$ that is generated by binomials $\partial^u - \lambda \partial^v$, where $u, v \in \mathbb{Z}^n$ and $\lambda \in \mathbb{C}$; such an ideal is \mathbf{A} -graded precisely when it is generated by binomials $\partial^u - \lambda \partial^v$ each of which satisfies either $\mathbf{A}u = \mathbf{A}v$ or $\lambda = 0$. Since $\mathbf{A}\mathbf{B}_i = 0, I(\mathbf{B}_i)$ is a \mathbf{A} -graded binomial ideal.

For the rest of the article, we let that for any graph G, \mathbf{A} is the matrix which we obtain in the Lemma 2.3. Also for simplicity we assume that the entries of \mathbf{A} are chosen from $\{0, -1, 1\}$.

Lemma 2.4. *I_L is a toral minimal prime ideal of \mathbf{A} -graded ideal $I(\mathbf{B}_i)$.*

Proof. We know that $\dim I_L = d$. By Corollary(2.1)[11], I_L is a minimal prime of $I(\mathbf{B}_i)$. Also by the previous Lemma, $\text{rank} \mathbf{A} = d$. Since L is a saturated lattice and for $I_L = I_{\rho, J}, J = \{1, \dots, n\}$, by Lemma(3.4)[4], I_L is toral.

□

Let $I_{\rho, J} = I_{\rho} + \langle \partial_j : j \notin J \rangle$ be the minimal prime of $I(\mathbf{B}_i)$, after row and column permutations, we have;

$$\begin{pmatrix} \mathbf{N} & \mathbf{B}_J \\ \mathbf{M} & \mathbf{0} \end{pmatrix}.$$

where \mathbf{M} is a mixed submatrix of \mathbf{B}_i of size $q \times p$ for some $0 \leq q \leq p \leq m$ [4]. The matrix \mathbf{M} has to satisfy another condition which is called irreducibility ([11], Definition 2.2 and Theorem 2.5). If $I(\mathbf{B})$ is a complete intersection, then only square matrices \mathbf{M} will appear in the block decompositions, by a result of Fischer and Shapiro [7].

Lemma 2.5. *Let $P \in \text{Min}I(\mathbf{B}_i), P \neq I_L$. Then P is Andean.*

Proof. Assume that decomposition of matrix \mathbf{B}_i for $I_{\rho, J} = P \neq I_L$, has the following form;

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{N} & \mathbf{B}_J \\ \mathbf{M} & \mathbf{0} \end{pmatrix}.$$

Since $I(\mathbf{B}_i)$ is a complete intersection ideal, \mathbf{M} is a square matrix. Let $I_{\rho, J}$ be a toral minimal prime and \mathbf{A}_J denotes the submatrix of \mathbf{A} whose columns are indexed by J . Since $\text{rank}(\ker_{\mathbb{Z}} \mathbf{A}_J) = d$, the matrix \mathbf{M} is an invertible matrix. The matrix \mathbf{M} corresponds to a directed cycle, then \mathbf{M} is not full rank, hence it isn't invertible. This is a contradiction, so $I_{\rho, J} = P$ is Andean.

□

Lemma 2.6. *$\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}$ for generic parameters are holonomic.*

Proof. Let $I_{\rho, J}$ be Andean. Also assume that the decomposition of \mathbf{B}_i has the following form;

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{N} & \mathbf{B}_J \\ \mathbf{M} & \mathbf{0} \end{pmatrix}.$$

We have $\det \mathbf{M} = 0$, so $\mathbb{C}\mathbf{A}_J \neq \mathbb{C}^d$. Therefore $\mathcal{Z}_{\text{Andean}}(I(\mathbf{B}_i)) \neq \mathbb{C}^d$, since $\mathcal{Z}_{\text{Andean}}(I(\mathbf{B}_i))$ is a union of finitely many integer translates of the subspaces $\mathbb{C}\mathbf{A}_J \subseteq \mathbb{C}^n$ for which there is an Andean associated prime $I_{\rho, J}$ [4]. Hence by Theorem(6.10)[4], the claim is proved.

□

The vertices v_1, \dots, v_t are called cellular cycle only if for all $i, 1 \leq i \leq t, N(v_i) \subseteq \{v_1, \dots, v_t\}$, where $N(v_i)$ is the neighborhood of the vertex v_i . We call the cellular cycle v_1, \dots, v_t minimal cellular cycle, if there isn't a vertex v_i such that, $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_t$ remains cellular cycle. we say that the graph G is partitioned to minimal cellular cycles if all minimal cellular cycles connect to each other by a vertex or a path.

Lemma 2.7. *The graph G can be partitioned to minimal cellular cycles.*

Proof. Let G_1 and G_2 be two cycles of G . If G_1 and G_2 are connected to each other by more than one path or one vertex, where every two paths between G_1 and G_2 have not common vertex, $G_1 \cup G_2$ include a cycle larger than G_1 and G_2 . In the same pattern, we review all cycles of G . We union all cycles which are connected to each other by two paths or more (every two paths between them have not common vertex), then the subgraphs Q_1, \dots, Q_t are formed. It is obvious that, $\forall i, j, i \neq j, 1 \leq i, j \leq t, Q_i$ and Q_j are connected to each other by a vertex or a path. Hence Q_1, \dots, Q_t are the minimal cellular partitions of G .

□

Proposition 2.8. *There are $k \in \mathbb{Z}$ and $\mathbf{w} \in \mathbb{Z}^d$ such that $P(k\mathbf{A} + \mathbf{w})$ is a reflexive polytope.*

Proof. The entries of the columns of the matrix \mathbf{A} are chosen from $\{-1, 1, 0\}$. First, let all entries of the columns of the matrix \mathbf{A} be zero or one. We must show that there is some $k \in \mathbb{Z}$ such that all vertices of the polytope $P(k\mathbf{A} + \mathbf{J})$ are primitive, where $\mathbf{J} = (-1, \dots, -1) \in \mathbb{Z}^d$. Suppose that $(1, \dots, 1) \in \mathbb{Z}^d$ is a column of \mathbf{A} , by choosing $k = 2$, the claim is proved. Otherwise all columns of the matrix have zero in their entries; in this case, let $k \gg 0$ such that the origin is in interior of $P(k\mathbf{A} + \mathbf{J})$. Now since every vertex of $P(k\mathbf{A} + \mathbf{J})$ has -1 in their entries, all of them are primitive.

Now let the entries of the columns of \mathbf{A} be $-1, 1$ and 0 . In this case, considering the position of placement of the polytope, like the previous case, we can choose suitable $k \in \mathbb{Z}$ and $\mathbf{w} \in \mathbb{Z}^d$ such that $P(k\mathbf{A} + \mathbf{w})$ is a reflexive polytope.

□

Proof. [Proof of Theorem 2.1] Let G be a directed graph, G_1, \dots, G_t the minimal cellular cycles of $G \setminus \{v_i\}$, and $\mathbf{B}_{G_1}, \dots, \mathbf{B}_{G_t}$ be, respectively, their incidence matrices. Also let $\mathbf{A}_{G_1}, \dots, \mathbf{A}_{G_t}$ be the matrices mentioned in Lemma 2.3, that;

$$\mathbf{A}_{G_i} \mathbf{B}_{G_i} = 0, \quad \forall i, 1 \leq i \leq t.$$

By an appropriate labeling we have:

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{B}_{G_1} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{B}_{G_2} & 0 & \dots & 0 \\ 0 & 0 & \mathbf{B}_{G_3} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{B}_{G_t} \end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{G_1} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{A}_{G_2} & 0 & \dots & 0 \\ 0 & 0 & \mathbf{A}_{G_3} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{A}_{G_t} \end{pmatrix}.$$

Note that the matrix \mathbf{A} may have some zero columns, which don't affect on volume of the matrix, so without loss of generality, we suppose that \mathbf{A} has no zero columns.

Assume that by Convention1.13, $P(\mathbf{A}_{G_1}), \dots, P(\mathbf{A}_{G_t})$ are the convex hulls of $\mathbf{A}_{G_1}, \dots, \mathbf{A}_{G_t}$ respectively. $P(\mathbf{A})$ is the free sum of $P(\mathbf{A}_{G_1}), \dots, P(\mathbf{A}_{G_t})$. By induction, it is enough to consider $t = 2$. By Proposition2.8, there are $k_1, k_2 \in \mathbb{Z}$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{Z}^d$ that $P(k_1\mathbf{A} + \mathbf{w}_1)$ and $P(k_2\mathbf{A} + \mathbf{w}_2)$ are reflexive polytopes. Put:

$$P(\mathbf{A}_{G_1}^*) = P(k_1\mathbf{A} + \mathbf{w}_1)$$

,

$$P(\mathbf{A}_{G_2}^*) = P(k_2\mathbf{A} + \mathbf{w}_2)$$

and

$$P(\mathbf{A}^*) = P(\mathbf{A}_{G_1}^*) \oplus P(\mathbf{A}_{G_2}^*).$$

By Theorem 2.11;

$$\text{Ehr}P(\mathbf{A}^*) = (1 - x)\text{Ehr}P(\mathbf{A}_{G_1}^*)\text{Ehr}P(\mathbf{A}_{G_2}^*).$$

That is

$$\text{vol}(\mathbf{A}^*) = \text{vol}(\mathbf{A}_{G_1}^*)\text{vol}(\mathbf{A}_{G_2}^*).$$

We know that;

$$\text{vol}(P + \boldsymbol{\alpha}) = \text{vol}P \text{ and } \text{vol}(cP) = c^{\dim P} \text{vol}P, \text{ where } \boldsymbol{\alpha} \in \mathbb{Z}^d, c \in \mathbb{Z}.$$

So

$$\text{vol}(\mathbf{A}) = \text{vol}(\mathbf{A}_{G_1})\text{vol}(\mathbf{A}_{G_2}).$$

□

Proposition 2.9. Let G be a cycle. If $-\beta \notin \mathcal{Z}_{\text{jump}}(I(\mathbf{B}_i))$, then $\text{rank}(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}) = 1$.

Proof. Because $d = 1$, the entries of the $1 \times n$ matrix \mathbf{A} are just 1. So $\text{vol}(\mathbf{A}) = 1$.

□

Proposition 2.10. Let G be a cycle with one diagonal, then $\text{rank}(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}) = 2$.

Proof. Let G be a cycle with one diagonal and m vertices. Without loss of generality, assume that the diagonal exits from $(m - 2)$ th vertex and enters to m th vertex. Consider the matrix \mathbf{B}_i , without loss of generality, put $i = m$. We have:

$$b_{11} = 1, \quad b_{i1} = 0, \quad 2 \leq i \leq m - 1.$$

By adding the second row to the first row, we have:

$$b_{13} = b_{23} = -1, \quad b_{33} = 1.$$

Now add the third row to first and second rows. By continuing this process, in reduced form of \mathbf{B}_m , we have:

$$b_{in-1} = -1, \quad 1 \leq i \leq m - 1,$$

$$b_{jn} = -1, \quad 1 \leq j \leq m - 2.$$

Then the matrix \mathbf{A} will have the following form:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Since the vectors $(1, 0)$, $(0, 1)$ and $(1, 1)$ contained in the set of the columns of \mathbf{A} ,

$$\text{rank}(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \beta)}) = \text{vol}(\mathbf{A}) = 2.$$

□

Theorem 2.11. Let G be a cycle with k diagonals that $k \geq 2$. Then the convex hull generated by the columns of the matrix \mathbf{A} has at least $2k + 2$ and at most $3k + 1$ vertices.

Proof. Let G has m vertices and \mathbf{B} be its incidence matrix. Without loss of generality, we choose the vertex v_i which has the most degree. There is a labeling that \mathbf{B}_i and its reduction form have the following forms:

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{M}_{m-1 \times m-1} \\ \mathbf{N}_{k+1 \times m-1} \end{pmatrix}$$

and

$$\mathbf{B}_i^{\text{red}} = \begin{pmatrix} \mathbf{I}_{m-1 \times m-1} \\ \mathbf{C}_{k+1 \times m-1} \end{pmatrix}.$$

The matrix \mathbf{N} has at most $2k$ nonzero entries distributed in at least k and at most $2k - 1$ rows. So the matrix \mathbf{C} has at least k and at most $2k - 1$ nontrivial nonequal rows. Hence the convex hull generated by the columns of the matrix \mathbf{A} has at least $2k + 2$ and at most $3k + 1$ vertices.

□

Corollary 2.12. Let G be a cycle with 2 diagonals, then;

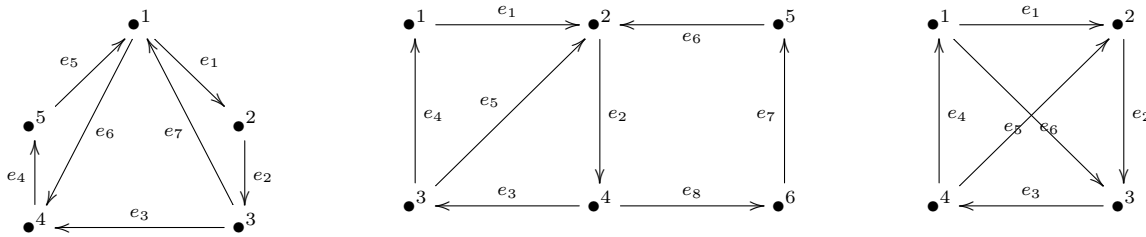
$$3 \leq \text{vol}(\mathbf{A}) \leq 5.$$

Proof. We know that for forming a cube, we need 8 vertices, but by Theorem 2.11 the convex hull generated by the columns of \mathbf{A} has at least 6 and at most 7 vertices. Then;

$$3 \leq \text{vol}(\mathbf{A}) \leq 5.$$

□

Example 2.1. Let G, H and K be the following graphs, respectively;



Also let the matrices \mathbf{A}, \mathbf{C} and \mathbf{D} be the corresponded matrices to the graphs G, H and K , respectively. One can compute:

$$\text{vol}(\mathbf{A}) = 3, \quad \text{vol}(\mathbf{C}) = 4 \quad \text{and} \quad \text{vol}(\mathbf{D}) = 5.$$

Finally considering Proposition 1.10, We can compute the volume of a convex hull corresponded to a cycle with $d - 1$ diagonals by the following algorithm.

Algorithm 2.13. Input: $d, \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subseteq \mathbb{Z}^d$. \mathbf{a}_i 's are the columns of the matrix \mathbf{A} .
Output: $\text{vol}(P(\mathbf{A}))$.

1. $c = \lfloor \frac{d}{2} \rfloor$
2. Choose a suitable integer k , such that $P(k\mathbf{A} + \mathbf{w})$ be a reflexive polytope.
3. Put $\mathbf{b}_i = k\mathbf{a}_i - \mathbf{w}$ and $E = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.
4. Consider all $\mathbf{b}_{1,s} = \sum \frac{\lambda_i}{k} \mathbf{b}_i \in \mathbb{Z}^d$ such that $\sum \lambda_i = k$.
5. Now let $E = \{\mathbf{b}_1, \dots, \mathbf{b}_{t_1}\}$
6. If $n = 1$ Finish. Otherwise go on.
7. Put $q = 2$.
8. Consider all $\mathbf{b}_{q,s} = \sum \frac{\lambda_i}{q} \mathbf{b}_i \in \mathbb{Z}^d$ such that $\sum \lambda_i = q$.
9. $E_q = \{\mathbf{b}_{q1}, \dots, \mathbf{b}_{qt_q}\}$.
10. If $n = q$ Finish. Otherwise put $q + 1 \rightarrow q$ and go 7.

$$11. \text{vol}(P(\mathbf{A})) = \frac{\sum_{b=0}^c (-1)^{c-b} \binom{d}{c-b} + (-1)^{d-1} \binom{d}{c+b+1}}{k^d}.$$

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