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Original Article

On the rank of the holomorphic solutions of PDE associated to directed graphs

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ABSTRACT: Let G be a directed graph with m vertices and n edges, $I(\mathbf{B})$ the binomial ideal associated to the incidence matrix $\mathbf{B}$ of the graph G, and I_L the lattice ideal associated to the columns of the matrix $\mathbf{B}$. Also let $\mathbf{B}_i$ be a submatrix of $\mathbf{B}$ after removing the ith column. In this paper it is determined that which minimal prime ideals of $I(\mathbf{B}_i)$ are Andean or toral. Then we study the rank of the space of solutions of binomial D-module associated to $I(\mathbf{B}_i)$ as $\mathbf{A}$-graded ideal, where $\mathbf{A}$ is a matrix that, $\mathbf{AB}_i = 0$. Afterwards, we define a miniaml cellular cycle and prove that for computing this rank it is enough to consider these components of G. We introduce some bounds for the number of the vertices of the convex hull generated by the columns of the matrix $\mathbf{A}$. Finally an algorthim is introduced by which we can compute the volume of the convex hull corresponded to a cycles with k diagonals, so by Theorem 2.1 the rank of $\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i),\boldsymbol{\beta})}$ can be computed.

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1. Introduction

The main object of study in this article is the binomial D-module, introduced in [3]. In the late 1980s, Gelfand, Graev and Zelevinsky introduced a class of systems of linear partial differential equations closely related to toric varieties [8]. These systems, called GKZ systems, or $\mathbf{A}$-hypergeometric systems, are constructed from a $d \times n$ integer matrix $\mathbf{A}$ of rank d and a complex parameter vector $\boldsymbol{\beta} \in \mathbb{C}^d$, and are denoted by $H_{\mathbf{A}}(\boldsymbol{\beta})$.

Convention 1.1. *Throughout the paper the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the following properties:*

1. $\boldsymbol{A} = (a_{ij}) \in \mathbb{Z}^{d\times n}$ *denotes an integer $d \times n$ matrix of rank d whose columns $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ all lie in a single open linear half-space of $\mathbb{R}^d$; equivalently, the cone generated by the columns of $\boldsymbol{A}$ is pointed (contains no lines), and all of the $\boldsymbol{a}_i$'s are nonzero. We also assume that $\mathbb{Z}\boldsymbol{A} = \mathbb{Z}^d$; that is, the columns of $\boldsymbol{A}$ span $\mathbb{Z}^d$ as a lattice.*
2. *Let $\boldsymbol{B} = (b_{jk}) \in \mathbb{Z}^{n\times m}$ be an integer matrix of full rank $m \leq n$. Assume that every nonzero element of the column-span of $\boldsymbol{B}$ over the integers $\mathbb{Z}$ is mixed, meaning that it has at least one positive and one negative entry; in particular, the columns of $\boldsymbol{B}$ are mixed. We write $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ for the rows of $\boldsymbol{B}$. Having chosen $\boldsymbol{B}$, we set $d = n - m$ and pick a matrix $\boldsymbol{A} \in \mathbb{Z}^{d\times n}$ whose columns span $\mathbb{Z}^d$ as a lattice, such that $\boldsymbol{AB} = 0$.*

Definition 1.2. *Let $\boldsymbol{A} \in \mathbb{Z}^{d\times n}$, so $\mathbb{Z}\boldsymbol{A} \subseteq \mathbb{Z}^d$ is a subgroup. A ring R is $\boldsymbol{A}$-graded if R is a direct sum of homogeneous components*

$$R = \oplus_{\alpha\in\mathbb{Z}\boldsymbol{A}} R_\alpha; \quad R_\alpha R_\beta \subseteq R_{\alpha+\beta}.$$

An ideal in an $\boldsymbol{A}$-graded ring is $\boldsymbol{A}$-graded if it is generated by homogeneous elements.

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Definition 1.3 (Definition 1.3 [3]). *For each $i \in \{1, \dots, d\}$, the ith Euler operator is;*

$$E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n,$$

where ∂_i is $\frac{\partial}{\partial_{x_i}}$. Given a vector $\boldsymbol{\beta} \in \mathbb{C}^d$, we write $\boldsymbol{E} - \boldsymbol{\beta}$ for the sequence $E_1 - \beta_1, \dots, E_d - \beta_d$. For an $\boldsymbol{A}$-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_{\boldsymbol{A}}(I, \boldsymbol{\beta})$, the left ideal $I + < E - \boldsymbol{\beta} >$ in the Weyl algebra D. The binomial D-module associated to I is $\frac{D}{H_{\boldsymbol{A}}(I,\boldsymbol{\beta})}$.

Given $\mathbf{A}$ as in Convention 1.1 , these are the left D-ideals $H_{\mathbf{A}}(I_{\mathbf{A}}, \boldsymbol{\beta})$, also denoted by $H_{\mathbf{A}}(\boldsymbol{\beta})$, where

$$I_{\mathbf{A}} =< \boldsymbol{\partial}^{\mathbf{u}} - \boldsymbol{\partial}^{\boldsymbol{v}} : \mathbf{A}.\mathbf{u} = \mathbf{A}.\mathbf{v} >\subseteq \mathbb{C}[\partial_1, \dots, \partial_n].$$

The $\mathbf{A}$-hypergeometric systems have many applications; for example, they arise naturally in the moduli theory of Calabi-Yau complete intersections in toric varieties, and (therefore) they play an important role in applications of mirror symmetry in mathematical physics [3].

Definition 1.4 (Definition 1.8 [3]). *Fix integer matrices $\boldsymbol{B}$ and $\boldsymbol{A}$ as in Convention1.1, and let $I(\boldsymbol{B})$ be the lattice basis ideal corresponding to this matrix, that is, the ideal in $\mathbb{C}[\boldsymbol{\partial}]$ generated by the binomials*

$$\prod_{b_{jk}>0} \partial_{x_j}^{b_{jk}} - \prod_{b_{jk}<0} \partial_{x_j}^{-b_{jk}}, \quad for \quad 1 \leq k \leq m.$$

The binomial Horn system with parameter $\boldsymbol{\beta}$ is the left ideal $H(\boldsymbol{B}, \boldsymbol{\beta}) = H_{\boldsymbol{A}}(I(\boldsymbol{B}), \boldsymbol{\beta})$ in the Weyl algebra $D = D_n$.

If $L \subseteq \mathbb{Z}^n$ is a sublattice, then the lattice ideal of L is

$$I_L =< \partial^{\mathbf{u}^+} - \partial^{\mathbf{u}^-} : u = u^+ - u^- \in L > .$$

Here and henceforth, u^+ has ith coordinate u_i if $u_i \geq 0$ and 0 otherwise. The vector $u^- \in \mathbb{N}^q$ is defined by $u^+ - u^- = u$, or, equivalently, $u^- = (-u)^+$. More general than I_L are the ideals

$$I_\rho =< \partial^{\mathbf{u}^+} - \partial^{\mathbf{u}^-} : u = u^+ - u^- \in L >$$

for any partial character $\rho : L \longrightarrow \mathbb{C}^*$ of $\mathbb{Z}^n$, which includes the data of both its domain lattice $L \subseteq \mathbb{Z}^n$ and the map to $\mathbb{C}^*$. The ideal I_ρ is prime if and only if L is a saturated sublattice of $\mathbb{Z}^n$, meaning that L equals its saturation

$$sat(L) = (\mathbb{Q}L) \cap \mathbb{Z}^n,$$

where $\mathbb{Q}L = \mathbb{Q} \otimes_{\mathbb{Z}} L$ is the rational vector space spanned by L in $\mathbb{Q}^n$. Every binomial prime ideal in $\mathbb{C}[\partial]$ has the form

$$I_{\rho,J} = I_\rho + < \partial_j : j \notin J >$$

for some saturated partial character ρ (i.e., whose domain is a saturated sublattice) and subset $J \subseteq \{1, \dots, n\}$ such that the binomial generators of I_ρ only involve variables $\partial_j, j \in J$.

Lemma 1.5. *(Lemma 3.4 [3]) Fix a partial character $\rho : L \to \mathbb{C}^*$ for a saturated sublattice $L \subseteq \mathbb{Z}^J \subseteq \mathbb{Z}^n$. Let $\mathcal{C}_{\rho,J}$ be an $\boldsymbol{A}$-graded binomial $I_{\rho,J}$-primary ideal. Then $L \subseteq \mathbb{Z}_J \cap \ker_{\mathbb{Z}}(\boldsymbol{A}) = \ker_{\mathbb{Z}}(\boldsymbol{A}_J)$, the Krull dimension satisfies $\dim(\mathbb{C}[\partial]/I_{\rho,J}) \geq rank(\boldsymbol{A}_J)$, and the following are equivalent.*

- *The Hilbert function $\mathbb{Z}\boldsymbol{A} \longrightarrow \mathbb{N}$ defined by $\alpha \longrightarrow \dim_{\mathbb{C}}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})_\alpha$ is bounded above.*
- *The homomorphism $\mathbb{Z}^J/L \to \mathbb{Z}\boldsymbol{A}_J \subseteq \mathbb{Z}^d$ is injective.*
- $L = \ker_{\mathbb{Z}}(\boldsymbol{A}_J)$.
- $\dim(\mathbb{C}[\partial]/I_{\rho,J}) = rank(\boldsymbol{A}_J)$.

When these conditions are satisfied, the module $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ and the lattice L are called toral, the ideal $I_{\rho,J}$ is called a toral prime, and $\mathcal{C}_{\rho,J}$ is called a toral (primary) component. When these conditions are not satisfied, substitute Andean for "toral" above.

In 2010 Dickenstein, Matusevich, and Miller presented and proved the following theorem [3]:

Theorem 1.6. *(Theorem 6.10 [3]) If $\mathcal{Z}_{Andean}(I) \neq \mathbb{C}^d$, then $H_{\boldsymbol{A}}(I, \boldsymbol{\beta})$ has minimal rank at $\boldsymbol{\beta}$ if and only if $-\boldsymbol{\beta}$ lies outside of the jump arrangement $\mathcal{Z}_{jump}(I)$, and this minimal rank is*

$$rank(\frac{D}{H_{\boldsymbol{A}}(I, \boldsymbol{\beta})}) = \sum_{I_{\rho,J} \quad toral \quad of \quad dim \quad d} \mu_{\rho,J}.vol\boldsymbol{A}_J.$$

Where $\mu_{\rho,J}$ be multiplicity of $I_{\rho,J}$ in I (or equivalently, in the primary component $\mathcal{C}_{\rho,J}$ of I) and $vol(\boldsymbol{A}_J)$ the volume of the convex hull of $\boldsymbol{A}_J$ and the origin, normalized so that a lattice simplex in the group $\mathbb{Z}\boldsymbol{A}_J$ generated by the columns of $\boldsymbol{A}_J$ has volume 1.

In order to obtain the necessary preliminaries for stating and proving the Proposition 2.8 and Theorem 2.1, we review some concepts as follow.
Let P be a lattice polytope of dimension d, i.e. a convex polytope in $\mathbb{R}^d$ whose vertices are elements of $\mathbb{Z}^d$ and whose affine span has dimension d, and P^o denote the interior of P. Given a positive integer n, the numerical functions $i(P, n)$ and $\bar{i}(P, n)$ are defined as follow:

$$i(P, n) = |nP \bigcap \mathbb{Z}^d|, \quad \bar{i}(P, n) = |n(P^o) \bigcap \mathbb{Z}^d|.$$

Here $nP = \{n\alpha : \alpha \in P\}$ and $|X|$ is the cardinality of a finite set X. Ehrhart in [7] stated the following properties:

1. $i(P, n)$ is a polynomial in n of degree d (and thus in particular $i(P, n)$ can be defined for every integer n);
2. $i(P, 0) = 1$;
3. $\bar{i}(P, n) = (-1)^d i(P, -n)$ for every integer $n \geq 0$.

Let

$$EhrP(x) = \sum_{n \geq 0} i(P, n)x^n = \frac{\sum_{j=0}^{d} h_j^\star x^j}{(1-x)^{d+1}},$$

denote the rational generating function for this polynomial, called the Ehrhart series of P.

Definition 1.7. *For two polytopes $P \subseteq \mathbb{R}^{d_P}$ and $Q \subseteq \mathbb{R}^{d_Q}$ of dimension d_P and d_Q, define the free sum to be*

$$P \oplus Q = conv\{(0_P \times Q) \bigcup (P \times 0_Q)\} \subseteq \mathbb{R}^{d_P + d_Q}.$$

Definition 1.8. *Let P be a lattice polytope in $\mathbb{R}^{d_P}$. The following set is called dual of P;*

$$P^\Delta = \{\boldsymbol{x} \in \mathbb{R}^{d_P} : \boldsymbol{x}.\boldsymbol{p} \leq 1 \quad for \quad all \quad \boldsymbol{p} \in P\}.$$

A lattice polytope whose dual is lattice polytope, called reflexive polytope.

Batyrev and Hibi in [1] and [10] respectively, proved the following lemma;

Lemma 1.9. *P is reflexive if and only if P is a lattice polytope with $0 \in P^o$ that satisfies one of the following (equivalent) conditions:*

1. *P^Δ is a lattice polytope.*
2. *$\bar{i}(P, n+1) = i(P, n)$ for all $n \in \mathbb{N}$, i.e. all lattice points in $\mathbb{R}^{d_P}$ sit on the boundary of some non-negative integral dilate of P.*
3. *$h_i^\star = h_{d_p - i}^\star$ for all i, where $h_i^\star$ is the ith coefficient in the numerator of the Ehrhart series for P.*

Proposition 1.10. *(Corollary 3.6. [9]) Let P be a d-dimensional reflexive polytope. Then*

$$vol(P) = \frac{1}{d!} \sum_{b=0}^{c} (-1)^{b+c} \left(\binom{d}{c-b} + (-1)^{d-1} \binom{d}{c+b+1} \right) i(P, b),$$

where $c := [\frac{d}{2}]$.

Braun in [2] proved the following theorem;

Theorem 1.11. *If P is a d_P-dimensional reflexive polytope in $\mathbb{R}^{d_P}$ and Q is a d_Q-dimensional lattice polytope in $\mathbb{R}^{d_Q}$ with $0 \in Q^o$, then*

$$Ehr((P \oplus Q)(x)) = (1-x)EhrP(x)EhrQ(x).$$

Definition 1.12. *Let P be a lattice polytope. A vertex of P is called primitive, if no lattice point lies strictly between the origin and the vertex.*

Convention 1.13. *The polytope corresponded to the matrix $\boldsymbol{A}$, is denoted by $P(\boldsymbol{A})$.*

Proposition 1.14. *Let P be a lattice polytope with 0 in its interior. P is reflexive if and only if each vertex is a primitive lattice point.*

Proof. It is straightforward corollary of Lemma 1.9. □

In this paper we consider a directed graph G whose vertices have both input and output edges, and let $\mathbf{B}$ be its incidence matrix, that is

$$b_{ij} = \begin{cases} -1 & e_j \ \ exits \ \ from \ \ v_i \\ 1 & e_j \ \ inters \ \ to \ \ v_i \\ 0 & otherwise \end{cases}$$

and $I(\mathbf{B})$ is defined as follow;

$$\mathrm{I}(\mathbf{B}) =< \partial^{\mathbf{u}_i^+} - \partial^{\mathbf{u}_i^-} | \mathbf{u}_i = \mathbf{u}_i^+ - \mathbf{u}_i^-, 1 \leq i \leq m, \mathbf{u}_i' s \ \ are \ \ the \ \ columns \ \ of \ \ \mathbf{B} > .$$

Suppose that $\mathbf{B}_i$ is a submatrix of $\mathbf{B}$ after removing the ith column. Assume that L is a lattice which generated by the column vectors of the matrix $\mathbf{B}$. It is shown that I_L is a toral minimal prime of $I(\mathbf{B}_i)$ and the others are Andean. Afterwards, we define a minimal cellular cycle and prove that for calculating the rank of $H(\mathbf{B}_i, \boldsymbol{\beta})$, it is enough to consider the minimal cellular cycle components of graph G. We introduce some bounds for the number of the vertices of the convex hull generated by the columns of the matrix $\mathbf{A}$. Finally an algorthim is introduced by which we can compute the volume of the convex hull corresponded to a cycles with k diagonals, so by Theorem 2.1 the rank of $\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i),\boldsymbol{\beta})}$ can be computed.

2. Main Results

By noting to Theorem 1.6 and using Lemma 2.4 and Lemma 2.5, for generic parameters, we have:

$$rank(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i), \boldsymbol{\beta})}) = vol(\mathbf{A}).$$

Hence by the following Theorem, for computing $rank(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i),\boldsymbol{\beta})})$, we consider the minimal cellular cycles, $G_1, \ldots, G_t$, calculate the volumes of corresponded convex hulls, and finally multiply all of them to compute the volume of $\mathbf{A}$.

Theorem 2.1. *Let G be a directed graph with m vertices and n edges. Suppose that $G_1, \ldots, G_t$ are the minimal cellular cycles of $G \backslash \{v_i\}$. Also let $\boldsymbol{A}_{G_1}, \ldots, \boldsymbol{A}_{G_t}$ be, respectively, the matrices corresponded to the minimal cellular cycles. For generic parameters, we have:*

$$rank(\frac{D}{H_{\boldsymbol{A}}(I(\boldsymbol{B}_i), \boldsymbol{\beta})}) = \prod_{i=1}^{t} vol(\boldsymbol{A}_{G_i}).$$

In the following, we will mention some necessary facts to prove this Theorem.

Lemma 2.2. *The lattice L is saturated.*

Proof. Since for all $i \in \{1, \ldots, m\}$,

$$-\mathbf{u}_i = \mathbf{u}_1 + \cdots + \mathbf{u}_{i-1} + \mathbf{u}_{i+1} + \cdots + \mathbf{u}_m,$$

without loss of generality, set $L =< \mathbf{u}_1, \ldots, \mathbf{u}_{m-1} >$. Let $b \in \mathbb{Z}$ and $\mathbf{Y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ such that $b\mathbf{Y} \in L$, that is:

$$\exists c_1, \ldots, c_{m-1} \in \mathbb{Z}; b\mathbf{Y} = c_1 \mathbf{u}_1 + \cdots + c_{m-1} \mathbf{u}_{m-1}.$$

Without loss of generality suppose in the first column of $\mathbf{B}$, the first entry is 1 and the last one is -1, then $by_1 = c_1$. Again without loss of generality suppose in the second column, the first entry is -1 and the second one is 1, we have:

$$by_2 = c_2 - c_1 = c_2 - by_1 \Longrightarrow c_2 = by_2.$$

Continuing this way, conclude that all of c_i's are multiplied by b, so

$$\mathbf{Y} = \frac{c_1}{b}\mathbf{u}_1 + \cdots + \frac{c_{m-1}}{b}\mathbf{u}_{m-1} \in L.$$

Then L is a saturated lattice. □

Lemma 2.3. *There exists a $d \times n$ matrix $\boldsymbol{A}$ of rank d such that for all i, $1 \leq i \leq m$, $\boldsymbol{A}\boldsymbol{B}_i = 0$.*

Proof. Since the lattice generated by the columns of $\mathbf{B}$ is the same as the lattices generated by the columns of each $\mathbf{B}_i$, $1 \leq i \leq m$, without loss of generality, we put:

$$L = < \mathbf{u}_1, \dots, \mathbf{u}_{m-1} > .$$

By Lemma 2.2, L is a saturated lattice, so I_L is a prime binomial ideal, hence there is some $d \times n$ matrix $\mathbf{A}$ that, $L = ker_{\mathbb{Z}}\mathbf{A}$. So for all i, $1 \leq i \leq m$, $\mathbf{A}\mathbf{B}_i = 0$. Now because $\mathbf{B}_i$ is full rank and $d = n - m + 1$, the matrix $\mathbf{A}$ is full rank too. □

The matrix $\mathbf{A}$ induces a $\mathbb{Z}^d$-grading of the polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n] = \mathbb{C}[\partial]$, called a $\mathbf{A}$-grading, by setting $deg(\partial_i) = a_i$, where a_i's are the columns of the matrix $\mathbf{A}$. Let I be a binomial ideal of $\mathbb{C}[\partial]$ that is generated by binomials $\partial^u - \lambda\partial^v$, where $u, v \in \mathbb{Z}^n$ and $\lambda \in \mathbb{C}$; such an ideal is $\mathbf{A}$-graded precisely when it is generated by binomials $\partial^u - \lambda\partial^v$ each of which satisfies either $\mathbf{A}u = \mathbf{A}v$ or $\lambda = 0$. Since $\mathbf{A}\mathbf{B}_i = 0$, $I(\mathbf{B}_i)$ is a $\mathbf{A}$-graded binomial ideal.
For the rest of the article, we let that for any graph G, $\mathbf{A}$ is the matrix which we obtain in the Lemma 2.3. Also for simplicity we assume that the entries of $\mathbf{A}$ are chosen from $\{0, -1, 1\}$.

Lemma 2.4. *I_L is a toral minimal prime ideal of $\boldsymbol{A}$-graded ideal $I(\boldsymbol{B}_i)$.*

Proof. We know that $dimI_L = d$. By Corollary(2.1)[11], I_L is a minimal prime of $I(\mathbf{B}_i)$. Also by the previous Lemma, $rank\mathbf{A} = d$. Since L is a saturated lattice and for $I_L = I_{\rho,J}$, $J = \{1, \dots, n\}$, by Lemma(3.4)[3], I_L is toral. □

Let $I_{\rho,J} = I_\rho + < \partial_j : j \notin J >$ be the minimal prime of $I(\mathbf{B}_i)$, after row and column permutations, we have;

$$\begin{pmatrix} \mathbf{N} & \mathbf{B}_J \\ \mathbf{M} & \mathbf{0} \end{pmatrix}.$$

where $\mathbf{M}$ is a mixed submatrix of $\mathbf{B}_i$ of size $q \times p$ for some $0 \leq q \leq p \leq m$ [3]. The matrix $\mathbf{M}$ has to satisfy another condition which is called irreducibility ([11], Definition 2.2 and Theorem 2.5). If I(B) is a complete intersection, then only square matrices M will appear in the block decompositions, by a result of Fischer and Shapiro [7].

Lemma 2.5. *Let $P \in MinI(\boldsymbol{B}_i), P \neq I_L$. Then P is Andean.*

Proof. Assume that decomposition of matrix $\mathbf{B}_i$ for $I_{\rho,J} = P \neq I_L$, has the following form;

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{N} & \mathbf{B}_J \\ \mathbf{M} & \mathbf{0} \end{pmatrix}.$$

Since $I(\mathbf{B}_i)$ is a complete intersection ideal, $\mathbf{M}$ is a square matrix. Let $I_{\rho,J}$ be a toral minimal prime and $\mathbf{A}_J$ denotes the submatrix of $\mathbf{A}$ whose columns are indexed by J. Since $rank(ker_{\mathbb{Z}}\mathbf{A}_J) = d$, the matrix $\mathbf{M}$ is an invertible matrix. The matrix $\mathbf{M}$ corresponds to a directed cycle, then $\mathbf{M}$ is not full rank, hence it isn't invertible. This is a contradiction, so $I_{\rho,J} = P$ is Andean. □

Lemma 2.6. *$\frac{D}{H_A(I(\boldsymbol{B}_i),\boldsymbol{\beta})}$ for generic parameters are holonomic.*

Proof. Let $I_{\rho,J}$ be Andean. Also assume that the decomposition of $\mathbf{B}_i$ has the following form;

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{N} & \mathbf{B}_J \\ \mathbf{M} & \mathbf{0} \end{pmatrix}.$$

We have $det\mathbf{M} = 0$, so $\mathbb{C}A_J \neq \mathbb{C}^d$. Therefore $\mathcal{Z}_{Andean}(I(\mathbf{B}_i)) \neq \mathbb{C}^d$, since $\mathcal{Z}_{Andean}(I(\mathbf{B}_i))$ is a union of finitely many integer translates of the subspaces $\mathbb{C}\mathbf{A}_J \subseteq \mathbb{C}^n$ for which there is an Andean associated prime $I_{\rho,J}$[3]. Hence by Theorem(6.10)[3], the claim is proved. □

The vertices $v_1, \dots, v_t$ are called cellular cycle only if for all i, $1 \leq i \leq t$, $N(v_i) \subseteq \{v_1, \dots, v_t\}$, where $N(v_i)$ is the neighborhood of the vertex v_i. We call the cellular cycle $v_1, \dots, v_t$ minimal cellular cycle, if there isn't a vertex v_i such that, $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_t$ remains cellular cycle. we say that the graph G is partitioned to minimal cellular cycles if all minimal cellular cycles connect to each other by a vertex or a path.

Lemma 2.7. *The graph G can be partitioned to minimal cellular cycles.*

Proof. Let G_1 and G_2 be two cycles of G. If G_1 and G_2 are connected to each other by more than one path or one vertex, where every two paths between G_1 and G_2 have not common vertex, $G_1 \cup G_2$ include a cycle larger than G_1 and G_2. In the same pattern, we review all cycles of G. We union all cycles which are connected to each other by two paths or more (every two paths between them have not common vertex), then the subgraphs $Q_1, \dots, Q_t$ are formed. It is obvious that, $\forall i, j, i \neq j, 1 \leq i, j \leq t$, Q_i and Q_j are connected to each other by a vertex or a path. Hence $Q_1, \dots, Q_t$ are the minimal cellular partitions of G. □

Proposition 2.8. *There are $k \in \mathbb{Z}$ and $\boldsymbol{w} \in \mathbb{Z}^d$ such that $P(k\boldsymbol{A} + \boldsymbol{w})$ is a reflexive polytope.*

Proof. The entries of the columns of the matrix $\mathbf{A}$ are chosen from $\{-1, 1, 0\}$. First, let all entries of the columns of the matrix $\mathbf{A}$ be zero or one. We must show that there is some $k \in \mathbb{Z}$ such that all vertices of the polytope $P(k\mathbf{A} + \mathbf{J})$ are primitive, where $\mathbf{J} = (-1, \dots, -1) \in \mathbb{Z}^d$. Suppose that $(1, \dots, 1) \in \mathbb{Z}^d$ is a column of $\mathbf{A}$, by choosing $k = 2$, the claim is proved. Otherwise all columns of the matrix have zero in their entries; in this case, let $k >> 0$ such that the origin is in interior of $P(k\mathbf{A} + \mathbf{J})$. Now since every vertex of $P(k\mathbf{A} + \mathbf{J})$ has -1 in their entries, all of them are primitive.
Now let the entries of the columns of $\mathbf{A}$ be $-1, 1$ and 0. In this case, considering the position of placement of the polytope, like the previous case, we can choose suitable $k \in \mathbb{Z}$ and $\mathbf{w} \in \mathbb{Z}^d$ such that $P(k\mathbf{A} + \mathbf{w})$ is a reflexive polytope. □

Proof of Theorem 2.1. Let G be a directed graph, $G_1, \dots, G_t$ the minimal cellular cycles of $G \backslash \{v_i\}$, and $\mathbf{B}_{G_1}, \dots, \mathbf{B}_{G_t}$ be, respectively, their incidence matrices. Also let $\mathbf{A}_{G_1}, \dots, \mathbf{A}_{G_t}$ be the matrices mentioned in Lemma 2.3, that;

$$\mathbf{A}_{G_i}\mathbf{B}_{G_i} = 0, \quad \forall i, 1 \leq i \leq t.$$

By an appropriate labeling we have:

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{B}_{G_1} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{B}_{G_2} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{B}_{G_3} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{B}_{G_t} \end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{G_1} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_{G_2} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{A}_{G_3} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{A}_{G_t} \end{pmatrix}.$$

Note that the matrix $\mathbf{A}$ may have some zero columns, which don't affect on volume of the matrix, so without loss of generality, we suppose that $\mathbf{A}$ has no zero columns.
Assume that by Convention 1.13, $P(\mathbf{A}_{G_1}), \dots, P(\mathbf{A}_{G_t})$ are the convex hulls of $\mathbf{A}_{G_1}, \dots, \mathbf{A}_{G_t}$ respectively. $P(\mathbf{A})$ is the free sum of $P(\mathbf{A}_{G_1}), \dots, P(\mathbf{A}_{G_t})$. By induction, it is enough to consider $t = 2$. By Proposition2.8, there are $k_1, k_2 \in \mathbb{Z}$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{Z}^d$ that $P(k_1\mathbf{A} + \mathbf{w}_1)$ and $P(k_2\mathbf{A} + \mathbf{w}_2)$ are reflexive polytopes. Put:

$$P(\mathbf{A}^\star_{G_1}) = P(k_1\mathbf{A} + \mathbf{w}_1)$$

,

$$P(\mathbf{A}^\star_{G_2}) = P(k_2\mathbf{A} + \mathbf{w}_2)$$

and

$$P(\mathbf{A}^\star) = P(\mathbf{A}^\star_{G_1}) \oplus P(\mathbf{A}^\star_{G_2}).$$

By Theorem2.11;

$$EhrP(\mathbf{A}^\star) = (1 - x)EhrP(\mathbf{A}^\star_{G_1})EhrP(\mathbf{A}^\star_{G_2}).$$

That is

$$vol(\mathbf{A}^\star) = vol(\mathbf{A}^\star_{G_1})vol(\mathbf{A}^\star_{G_2}).$$

We know that;

$$vol(P + \boldsymbol{\alpha}) = volP \quad and \quad vol(cP) = c^{dimP}volP, \quad where \quad \boldsymbol{\alpha} \in \mathbb{Z}^d, c \in \mathbb{Z}.$$

So

$$vol(\mathbf{A}) = vol(\mathbf{A}_{G_1})vol(\mathbf{A}_{G_2}).$$

□

Proposition 2.9. *Let G be a cycle. If $-\boldsymbol{\beta} \notin \mathcal{Z}_{jump}(I(\boldsymbol{B}_i))$, then $rank(\frac{D}{H_{\boldsymbol{A}}(I(\boldsymbol{B}_i),\boldsymbol{\beta})}) = 1$.*

Proof. Because $d = 1$, the entries of the $1 \times n$ matrix $\mathbf{A}$ are just 1. So $vol(\mathbf{A}) = 1$. □

Proposition 2.10. *Let G be a cycle with one diagonal, then $rank(\frac{D}{H_{\boldsymbol{A}}(I(\boldsymbol{B}_i),\boldsymbol{\beta})}) = 2$.*

Proof. Let G be a cycle with one diagonal and m vertices. Without loss of generality, assume that the diagonal exits from $(m-2)$th vertex and enters to mth vertex. Consider the matrix $\mathbf{B}_i$, without loss of generality, put $i = m$. We have:

$$b_{11} = 1, \quad b_{i1} = 0, \quad 2 \leq i \leq m-1.$$

By adding the second row to the first row, we have:

$$b_{13} = b_{23} = -1, \quad b_{33} = 1.$$

Now add the third row to first and second rows. By continuing this process, in reduced form of $\mathbf{B}_m$, we have:

$$b_{in-1} = -1, \quad 1 \leq i \leq m-1,$$

$$b_{jn} = -1, \quad 1 \leq j \leq m-2.$$

Then the matrix $\mathbf{A}$ will have the following form:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Since the vectors $(1,0), (0,1)$ and $(1,1)$ contained in the set of the columns of $\mathbf{A}$,

$$rank(\frac{D}{H_{\mathbf{A}}(I(\mathbf{B}_i),\boldsymbol{\beta})}) = vol(\mathbf{A}) = 2.$$

□

Theorem 2.11. *Let G be a cycle with k diagonals that $k \geq 2$. Then the convex hull generated by the columns of the matrix $\boldsymbol{A}$ has at least $2k+2$ and at most $3k+1$ vertices.*

Proof. Let G has m vertices and $\mathbf{B}$ be it's incidence matrix. Without loss of generality, we choose the vertex v_i which has the most degree. There is a labeling that $\mathbf{B}_i$ and it's reduction form have the following forms:

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{M}_{m-1\times m-1} \\ \mathbf{N}_{k+1\times m-1} \end{pmatrix}$$

and

$$\mathbf{B}_i^{red} = \begin{pmatrix} \mathbf{I}_{m-1\times m-1} \\ \mathbf{C}_{k+1\times m-1} \end{pmatrix}.$$

The matrix $\mathbf{N}$ has at most $2k$ nonzero entries distributed in at least k and at most $2k-1$ rows. So the matrix $\mathbf{C}$ has at least k and at most $2k-1$ nontrivial nonequal rows. Hence the convex hull generated by the columns of the matrix $\mathbf{A}$ has at least $2k+2$ and at most $3k+1$ vertices. □

Corollary 2.12. *Let G be a cycle with 2 diagonals, then;*

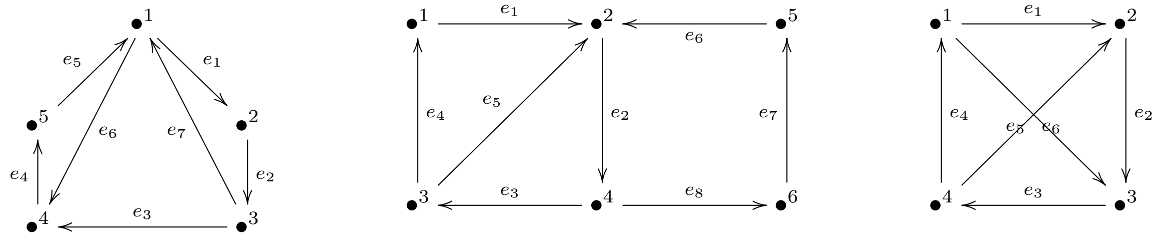
$$3 \leq vol(\boldsymbol{A}) \leq 5.$$

Proof. We know that for forming a cube, we need 8 vertices, but by Theorem 2.11 the convex hull generated by the columns of $\mathbf{A}$ has at least 6 and at most 7 vertices. Then;

$$3 \leq vol(\mathbf{A}) \leq 5.$$

□

Example 2.1. *Let G, H and K be the following graphs, respectively;*



Also let the matrices $\boldsymbol{A}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ be the corresponded matrices to the graphs G,H and K, respectively. One can compute:

$$vol(\boldsymbol{A}) = 3, \qquad vol(\boldsymbol{C}) = 4 \ \text{ and } \qquad vol(\boldsymbol{D}) = 5.$$

Finally considering Proposition 1.10, We can compute the volume of a convex hull corresponded to a cycle with $d-1$ diagonals by the following algorithm.

Algorithm 2.13. *Input: d, $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\} \subseteq \mathbb{Z}^d$. $\boldsymbol{a}_i$'s are the columns of the matrix $\boldsymbol{A}$.*
Output: $vol(P(\boldsymbol{A}))$.

1. $c = [\frac{d}{2}]$
2. *Choose a suitable integer k, such that $P(k\boldsymbol{A} + \boldsymbol{w})$ be a reflexive polytope.*
3. *Put $\boldsymbol{b}_i = k\boldsymbol{a}_i - \boldsymbol{w}$ and $E = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$.*
4. *Consider all $\boldsymbol{b}_{1,s} = \sum \frac{\lambda_i}{k} \boldsymbol{b}_i \in \mathbb{Z}^d$ such that $\sum \lambda_i = k$.*
5. *Now let $E = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{t_1}\}$*
6. *If $n = 1$ Finish. Otherwise go on.*
7. *Put $q = 2$.*
8. *Consider all $\boldsymbol{b}_{qs} = \sum \frac{\lambda_i}{q} \boldsymbol{b}_i \in \mathbb{Z}^d$ such that $\sum \lambda_i = q$.*
9. *$E_q = \{\boldsymbol{b}_{q1}, \ldots, \boldsymbol{b}_{qt_q}\}$.*
10. *If $n = q$ Finish. Otherwise put $q + 1 \to q$ and go 7.*
11. $vol(P(\boldsymbol{A})) = \dfrac{\sum_{b=0}^{c} (-1)^{c-b} \left(\binom{d}{c-b} + (-1)^{d-1} \binom{d}{c+b+1} \right) t_i}{k^d}.$

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