A linear-time algorithm to compute total $[1, 2]$-domination number of block graphs

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ABSTRACT: Let $G = (V, E)$ be a simple graph without isolated vertices. A set $D \subseteq V$ is a total $[1, 2]$-dominating set if for every vertex $v \in V$, $1 \leq |N(v) \cap D| \leq 2$. The total $[1, 2]$-domination problem is to determine the total $[1, 2]$-domination number $\gamma_{t[1, 2]}(G)$, which is the minimum cardinality of a total $[1, 2]$-dominating set for a graph $G$. In this paper, we present a linear-time algorithm to compute $\gamma_{t[1, 2]}(G)$ for a block graph $G$.

1. Introduction

All graphs considered here are simple, i.e., finite, undirected, and loop-less. For other graph theory terminology and notation not given here we refer to [10].

Let $G = (V, E)$ be a graph. The open neighborhood of a vertex $v \in V$ is the set of all vertices adjacent to $v$ and is denoted by $N(v)$. Similarly, the closed neighborhood of a vertex $v$ is $N[v] = N(v) \cup \{v\}$. In connected graph $G$, a vertex is called a cut-vertex of $G$ if its removal produces a disconnected graph. A block of a graph $G$ is a maximal connected induced subgraph of $G$ that has no cut-vertex. A block graph is a graph whose blocks are complete graphs. A subset $D \subseteq V$ is called a dominating set, if every vertex in $V$ is contained in $D$ or has a neighbor in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A total dominating set of a simple graph $G = (V, E)$ without isolated vertex, is a set $S \subseteq V$ such that every vertex is adjacent to a vertex in $S$. The minimum cardinality of a total dominating set is denoted by $\gamma_t(G)$. The minimum dominating set problem is an NP-hard problem [8, 7, 9].

A set $S \subseteq V$ is called a $[1, 2]$-set of $G$ if for each $v \in V - S$, $v$ is adjacent to at least one but not more than two vertices in $S$. The total $[1, 2]$-set is a set $S \subseteq V$ such that for each $v \in V$, $1 \leq |N(v) \cap S| \leq 2$. The total $[1, 2]$-domination number, denoted by $\gamma_{t[1, 2]}(G)$, is the minimum cardinality of a total $[1, 2]$-dominating set for a graph $G$. We note that in the problem total $[1, 2]$-dominating set, there are some graphs without any total $[1, 2]$-dominating sets, such as the graphs with isolated vertices.

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The concept of \([1,2]-\text{set and its variants such as total }[1,2]-\text{set and independent }[1,2]-\text{set are well studied by Chellali et al. in }[3,4]\. In [4], several open problems were proposed about \([1,2]-\text{set and total }[1,2]-\text{set. Some of these problems were solved in }[6,2,5]\. In this paper we provide a linear-time algorithm for finding the minimum cardinality of a total \([1,2]-\text{dominating set for a block graph }G\).

2. Algorithm for computing total \([1,2]-\text{domination number}

Our algorithm relies on a tree-like decomposition structure, which is called a refined cut-tree of a block graph. Let \(G\) be a block graph with \(t\) blocks \(B_1, B_2, \cdots, B_t\) and \(q\) cut-vertices \(v_1, v_2, \cdots, v_q\). The cut-tree of \(G\), denoted by \(T^C(V^C, E^C)\), is defined as \(V^C = \{B_1, B_2, \cdots, B_t, v_1, v_2, \cdots, v_q\}\) and \(E^C = \{(B_i, v_j) \mid v_j \in B_i, 1 \leq i \leq t, 1 \leq j \leq q\}\). The cut-tree of a block graph can be constructed in linear-time by the depth-first search algorithm [1]. For any block \(B_i\) of \(G\), the block-vertex \(B_i\) is defined as \(B_i = \{v \in B_i \mid v\) is not a cut-vertex\}, where \(1 \leq i \leq t\). We can refine the cut-tree \(T^C(V^C, E^C)\) as \(V^C = \{\tilde{B}_i, \cdots, \tilde{B}_t, v_1, \cdots, v_q\}\) and \(E^C = \{\tilde{(B}_i, v_j) \mid v_j \in \tilde{B}_i, 1 \leq i \leq t, 1 \leq j \leq q\}\). We notice that in the refined cut-tree of a block graph, a block-vertex can be empty.

A block graph \(G\) with 11 blocks \(B_1, B_2, \cdots, B_{11}\) and the corresponding refined cut-tree of \(G\) are shown in Figure 1.

\[\text{Figure 1: Block graph }G\text{ and the corresponding cut-tree of }G\]

To compute \(\gamma_{[1,2]}(G)\), we traverse \(T\) in the post order and during traversing, we compute \(m_i^-[v], m_i^+[v], S_0^- (u), S_{0,1}^- (u), S_{1,2}^- (u), S_0^+(u), S_{0,1}^+(u)\) and \(S^+(u)\) where \(i = 0, 1, 2, v\) is cut-vertex and \(u \in V(T)\) is block node.

- \(B\) as the set of all block nodes of \(T\).
- \(C\) as the set of all cut-vertices of \(G\).
- \(T_v\) as the subtree of \(T\) rooted at \(v\).
- \(G[T_v]\) as the subgraph of \(G\) which corresponding to \(T_v\).
- For the smallest \([1,2]-\text{set }S\text{ of }T_v\), every vertex in \(S\) is called \((S,v)\)-black, and \((S,v)\)-white otherwise. For simplicity, we use the terms black and white instead of \((S,v)\)-black and \((S,v)\)-white, respectively.
- For each block node \(u \in V(T)\) and \(v \in ch(u)\), depended on the color of \(v\) and the number of vertices which dominate \(v\), we define variables as bellow:
  - For \(u \in V(T), S_0^- (u)\) is the size of the smallest \([1,2]-\text{set of }G[T_v]\) that all children of \(u\) are white and they are not dominated.
  - For \(u \in V(T), S_{0,1}^- (u)\) is the size of the smallest \([1,2]-\text{set of }G[T_v]\) that all children of \(u\) are white and they are dominated at most once.
  - For \(u \in V(T), S_{1,2}^- (u)\) is the size of the smallest \([1,2]-\text{set of }G[T_v]\) that all children of \(u\) are white and they are dominated once or twice.
  - For \(u \in V(T), S_0^+(u)\) is the size of the smallest \([1,2]-\text{set of }G[T_v]\) that one child of \(u\) is black and it is dominated at most once. Moreover the other children of \(u\) are white and they are not dominated.
For $u \in V(T)$, $S^+(u)$ is the size of the smallest $[1, 2]$-set of $G[T_u]$ that one vertex of $ch(u)$ is black and it is dominated once or twice. Moreover the other children of $u$ are white and they are dominated at most once.

For $u \in V(T)$, $S^*(u)$ is the size of the smallest $[1, 2]$-set of $G[T_u]$ that two black vertices of $ch(u)$ are dominated at most once. Moreover the other children of $u$ are white and they are not dominated.

- For cut-vertex $v \in V(T)$, $m^+_i[v]$ is the size of the smallest $[1, 2]$-set of $G[T_v]$ that $v$ is white and dominated by $i$ other vertices of $G[T_v]$ for $i = 0, 1, 2$.

- For cut-vertex $v \in V(T)$, $m^-_i[v]$ is the size of the smallest $[1, 2]$-set of $G[T_v]$ that $v$ is black and dominated by $i$ other vertices of $G[T_v]$ for $i = 0, 1, 2$.

Now, we use a refined cut tree $T$ of a given block graph $G$ and dynamic programming method to compute total $[1, 2]$-domination number of $G$. The algorithm contains three step, Initializing step, Updating step and final step.

2.1. Initializing step:

Obviously, every leaf of cut-tree $T$ is a block node and it is not empty. Since variables for block nodes are based on color of its child, so in first step we begin our algorithm from pre-pendent node $v$ of $T$, that is a cut vertex of $G$. We initialize $m^+_i[v]$ and $m^-_i[v]$ for $i = 0, 1, 2$ and pre-pendent node $v$ of $T$ as bellow:

\[
\begin{align*}
    m^+_i[v] &= \begin{cases} 
        1 & \text{if } |ch(v)| = 1, \\
        \infty & \text{Otherwise,}
    \end{cases} \\
    m^-_i[v] &= \begin{cases} 
        2 & \text{if } |ch(v)| = 2, \\
        \infty & \text{Otherwise.}
    \end{cases}
\end{align*}
\]

2.2. Updating step:

In the post order traversal of $T$, for each non pre-pendent node, based on type of them which are a block or cut, we can consider the following cases:

2.3. Updating step for block nodes of $T$:

In this step, we define variables $S^0_0(v), S^0_1(v), S^1_0(v), S^1_2(v), S^0_{0,1}(u), S^1_{0,1}(u)$ and $S^*(u)$ for block node $u$ of $T$. These variables depend on the number of nodes in $ch(u)$.

Calculating $S^0_0(v)$:

All children of $u$ are white and they are not dominated. So:

\[
S^0_0(u) = \sum_{v \in ch(u)} m^0_0(v).
\]

Calculating $S^0_{0,1}(u)$:

All children of $u$ are white and they are dominated at most once. So:

\[
S^0_{0,1}(u) = \sum_{v \in ch(u)} \min \{m^0_0(v), m^1_0(v)\}.
\]

Calculating $S^1_1(u)$:

All children of $u$ are white and they are dominated at most once. So:

\[
S^1_2(u) = \sum_{v \in ch(u)} \min \{m^1_1(v), m^2_1(v)\}.
\]

Calculating $S^0_{0,1}(u)$:

One child $v_i \in ch(u)$ is black, it is dominated at most once and the other children of $u$ are white and they are not dominated. So:

\[
S^0_{0,1}(u) = \min_{v_i \in ch(u)} \{\min \{m^0_0(v_i), m^1_0(v_i)\} + \sum_{v \in ch(u), v \neq v_i} m^0_0(v)\}.
\]

Calculating $S^+_{0,1}(u)$:
One child \( v_i \in ch(u) \) is black and it is dominated once or twice. Moreover, the other children of \( u \) are white and they are dominated at most once. So:

\[
S^+_0(u) = \min_{v_i \in ch(u)} \{ \min \{ m_1^+(v_i), m_2^+(v_i) \} + \sum_{v \in ch(u), v \neq v_i} \min \{ m_0^-(v_i), m_1^-(v_i) \} \}.
\]

**Calculating** \( S^+(u) \):

Two vertices of \( v_i, v' \in ch(u) \) are black and they are dominated at most once. Moreover, the other children of \( u \) are white and they are not dominated. So:

\[
S^+(u) = \min_{v_i, v' \in ch(u)} \{ \min \{ m_0^+(v_i), m_1^+(v_i) \} + \min \{ m_0^+(v'_i), m_1^+(v'_i) \} + \sum_{v \in ch(u), v \neq v_i, v'} m_0^-(v_i) \}.
\]

2.4. **Updating step for cut vertex of** \( T \):

In this step, for \( i = 0, 1, 2 \) we define variables \( m_0^i(v) \) and \( m_1^i(v) \) for cut nodes \( v \) of \( T \).

**Calculating** \( m_0^i(v) \) when \( v \) is not a pre-pendent:

In this case, \( v \) is black so all the children of \( v \) and all children of \( ch(v) \) already have a black neighbor. Since \( v \) should not dominate by any vertices, So, all child \( u \in ch(v) \) must be white and they are dominated at most once.

\[
m_0^i(v) = 1 + \sum_{u \in ch(v)} S^{−}_0(u).
\]

**Calculating** \( m_1^i(v) \) when \( v \) is not a pre-pendent:

In this case, \( v \) is black and it is dominated once, so one of the following cases can occurs:

- In this case, exactly one vertex of block \( u_i \in ch(v) \) is black and other children of \( u_i \) have color white and are not dominated. For the other block \( u \in ch(v) \), all child are white and they are dominated at most once. We have:

\[
M_1^+ = 1 + \min_{u_i \in ch(v), \{u_i\} \neq 0} \{ S^{−}_0(u_i) + \sum_{u \in ch(v), u \neq u_i} S^{−}_0(u) \}.
\]

- For exactly one block \( u_i \in ch(v) \), one nodes of \( ch(u_i) \) and the black vertex dominate at most once. The other vertices of \( ch(u_i) \) are white and they are not dominated. In addition, for the other block \( u \in ch(v) \), all of their children are white and they are dominated at most once. We have:

\[
M_1^+ = \min_{u_i \in ch(v)} \{ S^+_0(u_i) + \sum_{u \in ch(v), u \neq u_i} S^{−}_0(u) \}.
\]

Minimum of \( M_1^+ \) and \( M_1^+ \) is the best value for \( m_1^+(v) \). So:

\[
m_1^+(v) = \min \{ M_1^+, M_1^+ \}.
\]

**Calculating** \( m_2^i(v) \) when \( v \) is not a pre-pendent:

In this case, \( v \) is black and it is dominated twice, so one of the following cases can occurs:

- There exist at least one vertex \( u_i \in ch(v) \) such that \( |u_i| = |ch(u_i)| = 1 \). In this case, vertex in block \( u_i \) is black, its child is black and it is not dominated. So we have:

\[
M_2^+ = 1 + \min_{u_i \in ch(v), |u_i|=|ch(u_i)|=1} \{ m_0^+(ch(u_i)) \} + \sum_{u \in ch(v), u \neq u_i} S^{−}_0(u).
\]

If there is not any node \( u_i \in ch(v) \) such that \( |u_i| = |ch(u_i)| = 1 \), then \( M_2^+ = \infty \).

- There exist at least one vertex \( u_i \in ch(v) \) such that \( |u_i| = 0 \) and \( |ch(u_i)| = 2 \). In this case, both children of \( ch(u_i) \) are black and it is not dominated. So we have:

\[
M_2^+ = \min_{u_i \in ch(v), |u_i|=0, |ch(u_i)|=2} \{ m_0^+(ch(u_i)) \} + \sum_{u \in ch(v), u \neq u_i} S^{−}_0(u).
\]

If there is not any node \( u_i \in ch(v) \) such that \( |u_i| = |ch(u_i)| = 1 \), then \( M_2^+ = \infty \).
• For exactly two blocks \( u_i, u_j \in ch(v) \), one vertex is black, so all children of \( ch(u_i) \) and \( ch(u_j) \) should be white and they are not dominated. For the other block \( u \in ch(v) \), all child must be white and they are dominated at most once. We have:

\[
M_2^{''''} = 2 + \min_{u_i, u_j \in ch(v)} \{ S_0^{-}(u_i) + S_0^{-}(u_j) + \sum_{u \in ch(v), u \neq u_i, u_j} S_0^{''}(u) \}.
\]

• For exactly one block \( u_i \in ch(v) \), one vertex is black, so all children of \( ch(u_i) \) should be white and they are not dominated. In addition, for exactly one block \( u_j \in ch(v) \), exactly one child is black and dominated at most once and the other child is white and not dominated. Also, all children other child \( u \in ch(v) \) should be white and they are dominated at most once.. We have:

\[
M_2^{''''} = 1 + \min_{u_i, u_j \in ch(v)} \{ S_0^{-}(u_i) + S_0^{''}(u_j) + \sum_{u \in ch(v), u \neq u_i, u_j} S_0^{'''}(u) \}.
\]

• For exactly two blocks \( u_i, u_j \in ch(v) \), exactly one child is black and dominated at most once and the other child is white and not dominated. Also, all children other child \( u \in ch(v) \) should be white and they are dominated at most once.. We have:

\[
M_2^{''''} = \min_{u_i, u_j \in ch(v)} \{ S_0^{-}(u_i) + S_0^{-}(u_j) + \sum_{u \in ch(v), u \neq u_i, u_j} S_0^{''''}(u) \}.
\]

Minimum of \( M_2^{''''}, M_2^{''''}, M_2^{''''}, M_2^{''''} \) and \( M_2^{''''} \) is the best value for \( m_1^{+}(v) \). So:

\[
m_1^{+}(v) = \min\{M_2^{''''}, M_2^{''''}, M_2^{''''}, M_2^{''''}\}.
\]

Calculating \( m_0^{-}(v) \) when \( v \) is not a pre-pendent

In this case, \( v \) is white and none of the child of \( v \) and \( ch(v) \) are not black. If \( v \) has a child like \( u \) that is a none empty block node, then vertices of \( u \) can not dominated by any vertices because \( v \) or \( ch(u) \) can only dominate \( u \). So, \( m_0^{-}(v) = \infty \) otherwise we have:

\[
m_0^{-}(v) = \sum_{u \in ch(v)} S_1^{-}(u).
\]

Calculating \( m_1^{-}[v] \) when \( v \) is not a pre-pendent

In this case, \( v \) is white and exactly one of the child of \( v \) or \( ch(v) \) are black.

• The node \( v \) has at least two children like \( u_1 \) and \( u_2 \) that are none empty block node. So vertices of \( u_1 \) or vertices of \( u_2 \) can not dominated and \( m_1^{-}(v) = \infty \).

• The node \( v \) has only one none empty child like \( u_1 \) and the other children are empty. So two cases appear:

1. One of the vertex of block node \( u_1 \) is black and all of children of \( u_1 \) are white and are dominated at most once.
2. All vertices of block node \( u_1 \) are white, exactly one child \( v_i \) of \( u_1 \) is black, the other are white. Moreover, \( v_i \) is dominated once or twice and white siblings of \( v_i \) are dominated at most once.

And we have:

\[
m_1^{-}(v) = \min\{1 + S_0^{''''}(u_1) + \sum_{u \in ch(v), u \neq u_1} S_0^{''}(u), S_0^{'''}(u_1) + \sum_{u \in ch(v), u \neq u_1} S_0^{''''}(u)\}
\]

• All children of \( v \) are empty. So, among all children of \( v \), there is exactly one node \( u_i \) such that \( u_i \) has only a black child the other is white. In other child \( u \neq u_i \), all nodes of \( ch(u) \) should be white and we have:

\[
m_1^{-}(v) = \min_{u_i \in ch(v)} \{ S_0^{''''}(u_i) + \sum_{u \in ch(v), u \neq u_i} S_0^{''''}(u) \}
\]

All children of \( v \) are empty. So, from exactly one child \( u_i \) of \( v \), one of the children is black and the others are white. In other child \( u \neq u_i \), all nodes of \( ch(u) \) should be white and we have:

\[
m_1^{-}(v) = \min_{u_i \in ch(v)} \{ S_0^{''''}(u_i) + \sum_{u \in ch(v), u \neq u_i} S_0^{''''}(u) \}
\]
Calculating $m_2^- [v]$ when $v$ is not a pre-pendent
In the last cases, we consider $v$ is white and exactly two children of $v$ or $ch(v)$ are black.

- The node $v$ has more than two none empty children, $m_2^- [v] = \infty$.
- The node $v$ has only two none empty children like $u_1$ and $u_2$ and the other children are empty. So, for $u_1$, $u_2$ and the other vertices one of the following cases appears:
  1. One of the vertex of block nodes $u_1$ and $u_2$ are black, all of children of them are white and are dominated at most once.
  2. One of the vertex of block node $u_1$ is black, all of children of it are white and are dominated at most once. Moreover, all of the vertices of block node $u_2$ are white, one of its child is black, the others are white and dominated at most once. ($u_1$ and $u_2$ can replace.)
  3. All of the vertices of block node $u_1$, $u_2$ are white, one of its child is black, the others are white and dominated at most once.

So we have:

$$m_2^-(v) = \text{Min}(2 + S^+_{0,1}(u_1) + S^-_{0,1}(u_2) + \sum_{u \in \text{ch}(v), u \neq u_1, u_2} S^-_{1,2},$$

$$1 + S^+_{0,1}(u_1) + S^-_{0,1}(u_2) + \sum_{u \in \text{ch}(v), u \neq u_1, u_2} S^-_{1,2},$$

$$1 + S^+_{0,1}(u_2) + S^-_{0,1}(u_1) + \sum_{u \in \text{ch}(v), u \neq u_1, u_2} S^-_{1,2},$$

$$S^+_{0,1}(u_1) + S^+_{0,1}(u_2) + \sum_{u \in \text{ch}(v), u \neq u_1, u_2} S^-_{1,2}.$$  

- The node $v$ has only one none empty child $u_1$. So, one of the following cases occurs for $u_1$:
  1. Two children of $u_1$ are black and the others are white and not dominated at most once.
  2. One vertex of block node $u_1$ and one of its children are black and the other children of $u_1$ are white and not dominated.
  3. One vertex of block node $u_1$ is black and all of children of $u_1$ are white and dominated at most once.
  4. All vertices of block node $u_1$ are white, one of its children are black and the other children of $u_1$ are white and not dominated.

For other child $u$ of ch($v$), there are exactly one node $u_i$ such that one child of it is black and the others are white. Obviously, all child of other siblings $u_i$ are white. So we have:

$$m_2^-(v) = \text{Min}(S^*(u_1) + \sum_{u \in \text{ch}(v), u \neq u_1} S^-_{1,2}(u_1),$$

$$1 + S^+_{0,1}(u_1) + \sum_{u \in \text{ch}(v), u \neq u_1} S^-_{1,2}(u_1),$$

$$1 + S^+_{0,1}(u_1) + \text{Min}_{u_i \in \text{ch}(v), u_i \neq u_1} \{ S^+_{0,1}(u_i) + \sum_{u \in \text{ch}(v), u \neq u_i, u_1} S^-_{1,2}(u) \},$$

$$S^+_{0,1}(u_1) + \text{Min}_{u_i \in \text{ch}(v), u_i \neq u_1} \{ S^+_{0,1}(u_i) + \sum_{u \in \text{ch}(v), u \neq u_i, u_1} S^-_{1,2}(u) \}.$$

- All children of $v$ are empty. So, from exactly two children $u_i, u_j$ of $v$, one of the children is black and the others are white. In other child $u \neq u_i, u_j$, all nodes of $ch(u)$ should be white and we have:

$$m_2^-(v) = \text{Min}_{u_i, u_j \in \text{ch}(v)} \{ S^+_{0,1}(u_i) + S^+_{0,1}(u_j) + \sum_{u \in \text{ch}(v), u \neq u_i, u_j} S^-_{1,2}(u) \}.$$
2.5. Final state:

Let \( r \) be the root of refined cut tree \( T \), \( r \) can be correspond to a cut vertex of \( G \) or a block of it. Depend on type of \( r \) one of the following cases appear:

1. The root \( r \) of \( T \) is a cut vertex of \( G \).

Since for \( i = 0, 1, 2 \) we compute \( m_i^+[v] \) and \( m_i^-[v] \) on a node of \( T \) that its corresponding vertex in \( G \) is a cut vertex, so we must choose best set among computed set of root \( r \). Note that \( r \) should be black or white and should be dominated by one or two vertices. It means that:

\[
M = \text{Min}\{m_1^+(r), m_2^+(r), m_1^-(r), m_2^-(r)\}.
\]

2. The root \( r \) of \( T \) is corresponding to a block of \( G \).

Since, we computed \( m^+ [v], m_0^+ [v], m_1^+ [v] \) and \( m_2^+ [v] \) for all vertices \( v \in ch(r) \). Based on the number of vertices in block \( r \) and the number of its child, one of the following cases appear:

(a) \( |r| = 0 \), so we have:

\[
M = \text{Min}\{S_{1,2}^-(r), S_{0,1}^+(r), S^*(r)\}.
\]

(b) \( |r| > 0 \), so we have:

\[
M = \text{Min}\{1 + S_0^+(r), 1 + S_{0,1}^-(r), S_{0,1}^+(r), S_{1,2}^-(r), S^*(r)\}.
\]

**Theorem 2.1.** The value \( M \) computed by Algorithm 1 for the block graph \( G \) is size of the smallest total \([1, 2]\)-set of \( G \) and is computed in linear-time.

**Proof.** The number of nodes in the refined cut tree \( T \) corresponding to block graph \( G \) is linear based on order of \( G \), i.e. \( n \). It is obvious that the algorithm traverses \( T \) once and is computed in linear-time \( O(n) \).

\[\square\]

**Algorithm 1 Total \([1, 2]\)-Dominating Set**

**Input:** A refined cut tree \( T \) of block graph \( G \).

1. **procedure Initializing step**

2. **procedure Updating step:** Depending on the type of non-pre-pendant nodes in the post order traversal of \( T \), one of the following procedures is selected:

3. **procedure Updating step for block nodes of \( T \):**

   Calculating \( S_0^- (u), S_0^+ (u), S_{0,1}^- (u), S_{1,2}^- (u), S_0^+ (u), S_{0,1}^+(u) \) and \( S^*(u) \).

4. **procedure Updating step for cut vertex of \( T \):**

   Calculating \( m_0^+ (v), m_1^+ (v), m_0^- (v), m_1^- (v) \) and \( m_2^- [v] \) when \( v \) is not a pre-pendant.

7. **procedure Final State**

8. if the root \( r \) of \( T \) is a cut vertex of \( G \) then \( M = \text{Min}\{m_1^+(r), m_2^+(r), m_1^-(r), m_2^-(r)\} \).

9. if the root \( r \) is a block and \( |r| = 0 \) then \( M = \text{Min}\{S_{1,2}^-(r), S_{0,1}^+(r), S^*(r)\} \).

10. if the root \( r \) is a block and \( |r| > 0 \) then

    \[
    M = \text{Min}\{1 + S_0^+(r), 1 + S_{0,1}^-(r), S_{0,1}^+(r), S_{1,2}^-(r), S^*(r)\}.
    \]

**Output:** Size of minimum total \([1, 2]\)-set of \( G \).

**References**


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