AUT Journal of Mathematics and Computing

AUT J. Math. Com., 1(2) (2020) 251-262
DOI: 10.22060/ajmc.2020.18629.1040

New directions in general fuzzy automata: a dynamic-logical view

Khadijeh Abolpour*[a], Mohammad Mehdi Zahedi[b,c], Marzieh Shamsizadeh[b,c]

[a]*Dept. of Math., Shiraz Branch, Islamic Azad University, Shiraz, Iran*
[b]*Dept. of Math., Graduate University of Advanced Technology, Kerman, Iran*
[c]*Dept. of Math., Behbahan Khatam Alanbia University of Technology, Behbahan, Iran*

ABSTRACT: In the current study, by a general fuzzy automaton we aim at showing a set of propositions related to a given automaton showing that the truth-values are depended on thestates, inputs and membership values of active states at time t. This new approach enables us to consider automata from a different point of view which is more close to logical treatment and helps us make estimations about the behavior of automaton particularly in a nondeterministic mode. The logic consists of propositions on the given GFA and its dynamic nature is stated by means of the so-called transition functor. This logic enables us to derive a certain relation on the set of states labeled by inputs. In fact, it is shown that if our set of propositions is large enough, this recovering of the transition relation is possible. Through a synthesis in the theory of systems, this study contributes to construct a general fuzzy automaton which realizes a dynamic process at least partially known to the user, which has been fully achieved in Theorem 3.6. Also, we study the theory of general fuzzy automata by using the concepts of operators. Such operators help us in the algebraic study of general fuzzy automata theory and provide a platform to use fuzzy topological therein. Further, a Galois connection is obtained between the state-transition relation on states and thetransition operators on propositions. To illustrate the proposed approach, the subject matter is more elaborated in detail through examples.

Review History:

Received:24 June 2020
Revised:09 August 2020
Accepted:10 August 2020
Available Online:01 September 2020

Keywords:

Dynamic Logic
General Fuzzy Automata
Proposition
Functor
Modal
Transition
Active State

1. Introduction

The concept of a finite automaton is well-known. Algebraic study of automata has been carried out by many authors in many forms (cf., e.g., [5, 10, 12]). After the introduction of fuzzy set in 1965 by Zadeh [36], a number of concepts in mathematics and other areas were fuzzified. Among the first such concepts was the concept of fuzzy automaton firstly proposed by Wee [35] and Santos [27], to deal with the notions such as vagueness and imprecision, frequently encountered in the study of natural languages. Further, Malik et. al. [16] introduced a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) and contributed greatly towards the algebraic study of fuzzy automata and fuzzy languages. M. Doostfatemeh and S. C. Kremer [9] used an extension of the notion of fuzzy automata and gave the concept of general fuzzy automata (for simplicity GFA). Their key motivation of introducing the notion general fuzzy automata was the insufficiency of the current literature to handle the applications which rely on fuzzy automata as a modeling tool, and assign membership values to active states of a fuzzy automaton. It will be interesting to see how the developed concepts and algorithms can be used in practice. A very interesting and challenging implication of our approach is that a zero-weight transition is possible and is different from no transition. A zero-weight transition may give rise to the activation of a successor due to the activation of its predecessor. A number of researchers have contributed to the growth of fuzzy automata theory.

**Corresponding author.*
E-mail addresses: Abolpor_kh@yahoo.com, zahedi_mm@kgut.ac.ir, shamsizadeh.m@gmail.com

Among these works, the work of Das [8] is towards the fuzzy topological characterization of a fuzzy automaton; the work of Jin and his coworkers [13] is towards the algebraic study of fuzzy automata based on po-monoids; the work of Kim, Kim and Cho [14] is towards the algebraic study of fuzzy automata theory; the work of Mockor [17, 18, 19] is towards the use of categorical concepts in the study of fuzzy automata theory; the work of Abolpour and Zahedi [1, 2, 3, 4] is towards the use of categorical concepts in the study of general fuzzy automata with membership values in different lattice structures; the work of Horry and Zahedi [11] is towards the use of fuzzy topologies for the study of a max-min general fuzzy automaton; the work of Qiu [23, 24, 25, 26] is towards the algebraic, topological and categorical study of fuzzy automata theory based on residuated lattices; the work of Tiwari and his coworkers [28, 29, 30, 31, 32, 33] is towards the algebraic, topological and categorical study of fuzzy automata; the work of Peeva [21, 22] is on the study of minimizing the states of fuzzy automata and its application to study pattern recognition; the work of Pal and the coworkers [20] is towards the study of fuzzy automaton based on residuated and co-residuated lattice. In previous studies on dynamic logic, all considerations are made on a physical system which is characterized by its states and transitions among the states. In fact, a dynamic logic B could assign the every automaton $\mathcal{A}$ not considering if $\mathcal{A}$ is deterministic or nondeterministic. This logic enables us to formulate observations on $\mathcal{A}$ in the form of composed propositions and, due to a transition functor T it captured the dynamic behavior of $\mathcal{A}$. There are formulated conditions under which the automaton $\mathcal{A}$ could be recovered by means of B and T. It is worth mentioning that, one of the most important topics in mathematics is searching for the relationship between logic and algebraic structures. Based on this, the theory of algebraic logic and ordered algebra has been developed. Thus, the main purpose of this work is to investigate the properties of one of the two structures based on the other. To this end, by establishing a relationship between logic structures and automata theory, we seek to access the properties of automata by using the properties of logic structures. In this regard, by a general fuzzy automaton we aim to demonstrate a set of propositions related to a given automaton showing that the truth-values are dependent on the states, inputs and membership values of active states at time t. This new approach allows us to consider automata from a different point of view which is more close to logical treatment and enables us to make estimations about the behavior of automaton particularly in a nondeterministic mode. In this respect, in the current study, investigating new directions in GFA based on a dynamic logical view, we realized that a general fuzzy automaton is a specific case of such a physical system and tried to rewrite it to the specific case. This study thus tries to present a new and general definition for fuzzy automata which not only encompassed all types of automata, including conventional fuzzy automata, but also several other computational paradigms. The present work, therefore, is an interdiseplinary study in which it investigated the relationship between a dynamic-logical structure and general fuzzy automata. Regarding this, some properties related to the dynamic-logical structures are compared and contrasted with those of general fuzzy automata. This approach can be compared with our model and the one suggested by [6] where an automaton can be characterized by an operator over a Hilbert space or it can be compared with the approach from [15] or [34].

2. preliminaries

In this section, the basic definitions and theorems used for the concepts in the next parts will be presented in detail.

Definition 2.1. *[9] A fuzzy set μ_Q defined on a set Q (discrete or continuous), is a function mapping each element of Q to a unique element of the interval [0,1].*

$$\mu_Q : Q \to [0,1]$$

Then, the fuzzy power set of Q denoted as $\tilde{P}(Q)$, is the set of all fuzzy subsets μ_Q, which can be defined on the set Q.

$$\tilde{P}(Q) = \{\mu_Q | \mu_Q : Q \to [0,1]\}$$

Definition 2.2. *[9] (Active state set) Knowing that the entered input prior for time t has been a_k, active states at time t are those states to which there is at least one transition on the input symbol a_k. Then, the fuzzy set of all active states at t (ordered pairs of states and their $mv's$) is called active state set at time t, and is denoted as $Q_{act}(t)$.*

Definition 2.3. *[9] A general fuzzy automaton (GFA) is considered as*

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2),$$

where (i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$, (ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$, *(v) $\omega : Q \to Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function. (vii) Function $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function. Function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition.*

With this definition, the process that occurs upon the transition from state q_i to q_j an input a_k is characterized by: $\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i,\, a_k, q_j))$.
It means that membership value (mv) of the state q_j at time $t+1$ is calculated by function F_1 utilizing both the membership value of q_i at time t and the weight of the transition.

There have been many options for the function $F_1(\mu, \delta)$. For instance, it can be $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$, $\dfrac{\mu+\delta}{2}$, or any other pertinent mathematical functions.
As it can be observed in the above mentioned, associated with each fuzzy transition, there exists a membership value (mv) in unit interval $[0,1]$. We identify this membership value as the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is designated as $\delta(q_i, a_k, q_j)$. Hereafter, we apply this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition; otherwise, it identifies the transition itself. The set of all transitions of a general fuzzy automaton $\tilde{F}$, is denoted as $\Delta_{\tilde{F}}$. However, whenever it is understood we remove the subscript, and write simply Δ. Concerning this, we say that Δ is a state-transition relation and it is regarded as a dynamics of $\tilde{F}$. On the other hand, we regularly formulate certain propositions on an automaton $\tilde{F}$ and draw conclusions from the behavior of $\tilde{F}$ in the present or in the future.
(viii) $F_2 : [0,1]^* \to [0,1]$, is called multi-membership resolution function. The multi-membership resolution function determines the multi-membership active states and allocates a single membership value to them.
We let $Q_{act}(t_i)$ be the set of all active states at time t_i, $\forall_i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) | \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}$, $\forall i \geq 1$. Since $Q_{act}(t_i)$ is a fuzzy set, to demonstrate that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subseteq Domain(Q_{act}(t_i))$; henceforth. We simply specify them by: $q \in Q_{act}(t_i)$ and $T \subseteq Q_{act}(t_i)$.

Definition 2.4. *[7] Let S be a non-empty set. Every subset $R \subseteq S \times S$ is called a relation on S and we declare that the couple (S, R) is a transition frame.*

Definition 2.5. *[7] A mapping f is called order - preserving or monotone if $a, b \in A$ and $a \leq b$ together imply $f(a) \leq f(b)$ and order-reflecting if $a, b \in A$ and $f(a) \leq f(b)$ together imply $a \leq b$. A bijective order-preserving and order-reflecting mapping $f : A \to B$ is called an isomorphism and then we state that the partially ordered sets $(A; \leq)$ and $(B; \leq)$ are isomorphic.*

Definition 2.6. *[7] Let $(A; \leq)$ and $(B; \leq)$ be partially order sets. A mapping $f : A \to B$ is called residuated if there is a mapping $g : B \to A$ so that $f(a) \leq b$ if and only if $a \leq g(b)$ for all $a \in A$ and $b \in B$. In this case, we state that f and g form a residuated pair or that the pair (f, g) is a (monotone) Galois connection. The role of Galois connections is indispensable for our constructions.*

Definition 2.7. *[7] If a partially ordered set $\mathbf{A}$ has both a bottom and a top element, it will be called bounded; the pertinent notation for a bounded partially ordered set is $(A; \leq, 0, 1)$. Let $(A; \leq, 0, 1)$ and $(B; \leq, 0, 1)$ be bounded partially ordered sets. A morphism $f : A \to B$ of bounded partially ordered sets is an order, top element and bottom element preserving map.*

Observation 2.8. *[7] Let $\mathbf{A}$ and $\mathbf{M}$ be bounded partially ordered sets, and $h_s : A \to M$, $s \in S$, morphisms of bounded partially ordered sets. The conditions are equivalent:*
(i) $\forall s \in S \;\; h_s(a) \leq h_s(b) \;\Rightarrow a \leq b$ for any element $a, b \in A$;
(ii) The map $i_A^S : A \to M^S$ defined by $i_A^S(a) = (h_s(a))_{s \in S}$ for all $a \in A$ is order-reflecting.
Then, we declare that $\{h_s : A \to M; s \in S\}$ is a full set of order-preserving mappings concerning M. Note that in this situation we may specify $\mathbf{A}$ with a bounded subposets of $\mathbf{M}^S$ because i_A^S is an order reflecting morphism alias embedding of bounded partially ordered sets. For any $s \in S$ and any $p = (p_s)_{s \in S}$ we indicate by $p(s)$ the s-th projection p_s. Note that $i_A^S(a)(s) = h_s(a)$ for all $a \in A$ and all $s \in S$.

Definition 2.9. *[7] Let $\boldsymbol{A} = (A; \leq, 0, 1)$ and $\boldsymbol{B} = (B; \leq, 0, 1)$ be bounded posets with a full set S of morphisms of bounded posets into a non-trivial complete lattice $\boldsymbol{M}$. We may assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are bounded subposets of $\boldsymbol{M}^S$. Let $P : A \to B$ and $T : B \to A$ be morphisms of posets. Let us define the relations*

$$R_T = \{(s, t) \in S \times S | (\forall b \in B)(s(T(b)) \leq t(b))\}$$

and

$$R^P = \{(s, t) \in S \times S | (\forall a \in A)(s(a) \leq t(P(a)))\}.$$

Lemma 2.10. *[7] Let $\mathbf{M}$ be a non-trivial complete lattice and S a non-empty set so that $\mathbf{A}$ and $\mathbf{B}$ are bounded subposets of $\mathbf{M}^S$. Let $P : A \to M^S$ and $T : B \to M^S$ be morphisms of posets so that, for all $a \in A$ and all $b \in B$,*

$$P(a) \leq b \Leftrightarrow a \leq T(b).$$

(a) If $P(A) \subseteq B$ then $R_T \subseteq R^P$.
(b) If $T(B) \subseteq A$ then $R^P \subseteq R_T$.
(c) If $P(A) \subseteq B$ and $T(B) \subseteq A$ then $R_T = R^P$.

3. Construction of general fuzzy automata by dynamic logical view

One of the most important indicators in general fuzzy automaton is the existence of active state membership values based on which we obtain different results. Therefore, we use active state membership values to construct the logic structure. As a result, both active state membership values and the logical structure created by them are discussed in this article. This section aims at deriving the logic B which is a set of propositions about the general fuzzy automaton $\tilde{F}$ formulated by the observer and constructing an ordered algebra structure on B. If we fix an input $a_k \in \Sigma$ at time t_i the proposition $\alpha|_{a_k}$ can be computed by $\mu^{t_i}(q_i)$ if the general fuzzy automaton $\tilde{F}$ is in the state q_i at time t_i otherwise $\alpha|_{a_k}$ is 0 if $\tilde{F}$ is not in the active state q_i. Thus, for each state $q_i \in Q$ we can assess the truth value of $\alpha|_{a_k}$, it is indicated by $\alpha|_{a_k}(q_i)$. As explained above, $\alpha|_{a_k}(q_i) \in [0, 1]$. We can establish the order $\leq$ on B as follows: for $\alpha, \beta \in B$, $\alpha \leq \beta$ if and only if $\alpha(q_i) \leq \beta(q_i)$ for all $q_i \in Q$. One can instantly check that the contradiction, i.e., the proposition with constant truth value 0, is the least element and the tautology, i.e., the proposition with the constant truth value 1 is the greatest component of the partially order set $(B; \leq)$. Note that any component ith of 1 is the maximum membership values of active states at time t_i, for any $i \geq 0$. This fact will be stated by the notation $\mathbf{B} = (B; \leq, 0, 1)$ for the bounded partially order set of proposition about the general fuzzy automaton $\tilde{F}$. Every automaton $\tilde{F}$ will be identified with the triple (B, Σ, Q), where B is the set of propositions about $\tilde{F}$, Σ is the set of possible inputs and Q is the set of states on $\tilde{F}$. In what follows, the truth-values of our logic $\mathbf{B}$ will be considered to be from the complete lattice $\mathbf{M} = ([0, 1]; \leq, 0, 1)$. Thus $\mathbf{B}$ will be bounded subposet of $\mathbf{M}^Q$ for the complete lattice $\mathbf{M}$ of truth-values.
We are given a set of labeled transitions $\Delta \subseteq Q \times \Sigma \times Q$ so that for an input $a_k \in \Sigma$, $\tilde{F}$ can go from q_i to q_j provided $\delta(q_i, a_k, q_j) \in \Delta$. As in the following, let $\mathbf{M} = ([0, 1]; \leq, 0, 1)$ be a bounded partially ordered set and the bounded subposets $\mathbf{A} = (A; \leq, 0, 1)$ and $\mathbf{B} = (B; \leq, 0, 1)$ of $\mathbf{M}^Q$ will stand for the possibly different logics of propositions pertaining to our automaton $\tilde{F}$, a corresponding set of states Q, and a state-transition relation Δ on Q. The operator $T_\delta : B \to (M^Q)^\Sigma$ will prescribe to a proposition $b \in B$ about $\tilde{F}$ a new proposition $T_\delta(b) \in (M^Q)^\Sigma$ so that the truth value of $T_\delta(b)$ in state $q_m \in Q$ is the greatest truth value that is smaller than or equal to the corresponding truth values of b in all states of $Q_{succ}(q_m, a_k)$. If there exists no such state, the truth value of $T_\delta(b)$ in state q_m will be 1. Similarly, the operator $P_\delta : A \to (M^Q)^\Sigma$ will prescribe to a proposition $a \in A$ about $\tilde{F}$ a new proposition $P_\delta(a) \in (M^Q)^\Sigma$ so that the truth value of $P_\delta(a)$ in state $q_m \in Q$ is the smallest truth value that is greater than or equivalent to the corresponding truth values of a in all states of $Q_{pred}(q_m, a_k)$. If there is no such state, the truth value of $P_\delta(a)$ in state q_m will be 0. Reflect on a complete lattice $\mathbf{M} = ([0, 1]; \leq, 0, 1)$ and let $\mathbf{A} = (A; \leq, 0, 1)$ and $\mathbf{B} = (B; \leq, 0, 1)$ be bounded partially ordered sets with a full set Q of morphisms of bounded partially ordered sets into a non-trivial complete lattice $\mathbf{M}$. We may assume that $\mathbf{A}$ and $\mathbf{B}$ are bounded subposets of $\mathbf{M}^Q$. Additionaly, let (Q, Δ) be a transition frame. We can define mappings $P_\delta : A \to (M^Q)^\Sigma$ and $T_\delta : B \to (M^Q)^\Sigma$ as follows:

For all $b \in B$ and $q_m \in Q$, $a_k \in \Sigma$,

$$T_{\delta_{a_k}}(b)(q_m) = \wedge_M \{b(q_j) | q_j \in Q_{succ}(q_m, a_k)\}, \quad (*)$$

where

$$Q_{succ}(q_m, a_k) = \{q_j | \delta(q_m, a_k, q_j) \in \Delta\},$$

and for all $a \in A$

$$P_{\delta_{a_k}}(a)(q_m) = \vee_M \{a(q_j) | q_j \in Q_{pred}(q_m, a_k)\}, \quad (**)$$

where

$$Q_{pred}(q_m, a_k) = \{q_j | \delta(q_j, a_k, q_m) \in \Delta\}.$$

Then we state that $T_\delta(P_\delta)$ is an upper transition functor (lower transition functor) constructed through the transition frame (Q, Δ), respectively. We signify that T_δ is an order-preserving map so that $T_\delta(1) = 1$ and correspondingly, P_δ is an order-preserving map such that $P_\delta(0) = 0$.

In order to illustrate our approach, we characterize the following example.

Example 3.1. *Consider the GFA in Figure 1. It is specified as:* $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, *where* $Q = \{q_0, q_1, q_2\}$ *is the set of states,* $\Sigma = \{a, b\}$ *is the set of input symbols,* $\tilde{R} = \{(q_0, 1)\}$, $Z = \emptyset$ *and* ω *is not applicable. If we choose* $F_1(\mu, \delta) = \delta, F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$, *then we have:*

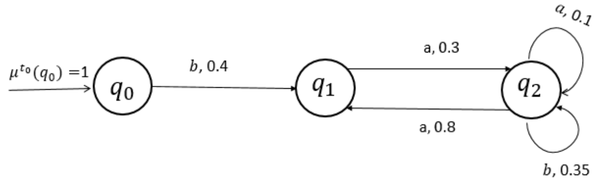

Figure 1: The GFA of Example 3.1

$$\begin{aligned}
\mu^{t_0}(q_0) &= 1\\
\mu^{t_1}(q_1) &= F_1(\mu^{t_0}(q_0), \delta(q_0, b, q_1)) = \delta(q_0, b, q_1) = 0.4,\\
\mu^{t_2}(q_2) &= F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = \delta(q_1, a, q_2) = 0.3,\\
\mu^{t_3}(q_1) &= F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_1)) = \delta(q_2, a, q_1) = 0.8,\\
\mu^{t_3}(q_2) &= F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_2)) = \delta(q_2, a, q_2) = 0.1,\\
\mu^{t_4}(q_2) &= F_1(\mu^{t_3}(q_2), \delta(q_2, b, q_2)) = \delta(q_2, b, q_2) = 0.35.
\end{aligned}$$

The set $B = \{0, s_0, s_1, s_2, s_0', s_1', s_2', 1\}$ *of possible propositions* B *about the automaton* $\tilde{F}$ *is as follows:*

Table 1: Active states and their membership values (mv) at different times in Example 3.1 upon input string "ba^2b"

time	t_0	t_1	t_2	t_3		t_4
input	$\wedge$	b	a	a		b
$Q_{act}(t_i)$	q_0	q_1	q_2	q_1	q_2	q_2
mv	1	0.4	0.3	0.8	0.1	0.35

-0 means that the GFA is not in active states of Q,
-s_0 *means that the GFA is in active state* q_0,
-s_1 *means that the GFA is in active state* q_1,
-s_2 *means that the GFA is in active state* q_2,
-s_0' *means that the GFA is either in active state* q_1 *or in the active state* q_2,
-s_1' *means that the GFA is either in active state* q_0 *or in the active state* q_2,
-s_2' *means that the GFA is either in active state* q_0 *or in the active state* q_1,
-1 means that the GFA is in at least one active state of Q.
We may have B *with the algebra* $[0, 1]^Q$ *as follows:* $0 = (0, 0, 0)$, $s_0 = (1, 0, 0)$, $s_1 = (0, 0.8, 0)$, $s_2 = (0, 0, 0.35)$, $s_0' = (0, 0.8, 0.35)$, $s_1' = (1, 0, 0.35)$, $s_2' = (1, 0.8, 0)$, $1 = (1, 0.8, 0.35)$.
Here, $\alpha(q_i)$ *is the maximum membership values of active states at time* t_i *for any* $i \geq 0$. *We have* $\delta_a = \{(q_1, q_2), (q_2, q_1), (q_2, q_2)\}$ *and* δ_b=$\{(q_0, q_1), (q_2, q_2)\}$, *then* $\Delta = \{(q_0, q_1), (q_1, q_2), (q_2, q_1), (q_2, q_2)\}$. *Using our formulas (*) and (**), we can obtain the upper transition functors* $T_{\delta_a}, T_{\delta_b} : B \to [0, 1]^Q$ *and the lower transition functors* $P_{\delta_a}, P_{\delta_b} : B \to [0, 1]^Q$ *as follows:*

$$\begin{array}{ll}
T_{\delta_a}(0) = 0, & P_{\delta_a}(0) = 0,\\
T_{\delta_a}(0) = s_0, & P_{\delta_a}(s_0) = 0,\\
T_{\delta_a}(s_2) = s_1, & P_{\delta_a}(s_1) = s_2,\\
T_{\delta_a}(s_0') = s_2, & P_{\delta_a}(s_2) = s_0',\\
T_{\delta_a}(s_0') = s_0', & P_{\delta_a}(s_0') = s_0',\\
T_{\delta_a}(s_0') = s_1', & P_{\delta_a}(s_1') = s_0',\\
T_{\delta_a}(s_2) = s_2', & P_{\delta_a}(s_2') = s_2,\\
T_{\delta_a}(1) = 1, & P_{\delta_a}(1) = 1,
\end{array}$$

$$\begin{array}{ll}
T_{\delta_b}(0) = 0, & P_{\delta_b}(0) = 0, \\
T_{\delta_b}(s_1) = s_0, & P_{\delta_b}(s_0) = s_1, \\
T_{\delta_b}(0) = s_1, & P_{\delta_b}(s_1) = 0, \\
T_{\delta_b}(s_2) = s_2, & P_{\delta_b}(s_2) = s_2, \\
T_{\delta_b}(s_2) = s_0', & P_{\delta_b}(s_0') = s_2. \\
T_{\delta_b}(s_0') = s_1', & P_{\delta_b}(s_1') = s_0', \\
T_{\delta_b}(s_1) = s_2', & P_{\delta_b}(s_2') = s_1, \\
T_{\delta_b}(1) = 1, & P_{\delta_b}(1) = 1.
\end{array}$$

E.g. $T_{\delta_a}(s_2) = s_1$ *meaning that if the GFA is in active state* q_1*, when entering input* a*, will change to* q_2 *and* $P_{\delta_a}(s_2) = s_0'$ *meaning that if the GFA is in active state* q_2*, when entering* a*, will change to* q_1 *or* q_2*.*

Let us define the relations

$$\Delta_{T_\delta} = \{(q_i, q_j) \in Q \times Q | \forall b \in B, T_\delta(b)(q_i) \leq b(q_j)\},$$
$$\Delta^{P_\delta} = \{(q_i, q_j) \in Q \times Q | \forall a \in A, a(q_i) \leq P_\delta(a)(q_j)\}.$$

In this situation, we assert that Δ is recoverable from T_δ or that Δ is recoverable from P_δ. We claim that Δ is recoverable if it is recoverable both from T_δ and P_δ.

Let us consider a general fuzzy automaton $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$. Clearly, Δ can be written in the following form

$$\Delta = \bigcup_{a_k \in \Sigma} \delta_{a_k}$$

where $\delta_{a_k} \subseteq Q \times Q$ for all $a_k \in \Sigma$. Hence, for all $a_k \in \Sigma$, using our formulas $(*)$ and $(**)$, we obtain the upper transition functor $T_{\delta_{a_k}} : B \to M^Q$ and the lower transition functor $P_{\delta_{a_k}} : B \to M^Q$. It follows that we have functors $T_\delta = (T_{\delta_{a_k}})_{a_k \in \Sigma} : B \to (M^Q)^\Sigma$ and $P_\delta = (P_{\delta_{a_k}})_{a_k \in \Sigma} : B \to (M^Q)^\Sigma$. We state that T_δ is the labeled upper transition functor constructed by means of $\tilde{F}$ and P_δ is the labeled lower transition functor constructed by means of $\tilde{F}$. Note that any mapping $T : B \to (M^Q)^\Sigma$ corresponds to a mapping $\tilde{T} : \Sigma \times B \to M^Q$ so that, for all $a_k \in \Sigma$, $T = (\tilde{T}(a_k, -))_{a_k \in \Sigma}$.
Hence, T_δ and P_δ will play the role of our transition functor. Now, let $P = (P_{a_k})_{a_k \in \Sigma} : B \to (M^Q)^\Sigma$ and $T = (T_{a_k})_{a_k \in \Sigma} : B \to (M^Q)^\Sigma$ be morphisms of partially ordered sets.
For all $a_k \in \Sigma$, let $\Delta^{P_{a_k}}$ be the lower P_{a_k}-induced relation by $\mathbf{M}$ and $\Delta_{T_{a_k}}$ be the upper T_{a_k}-induced relation by $\mathbf{M}$. Then $\Delta^P = \bigcup_{a_k \in \Sigma} \Delta^{P_{a_k}}$ is called the lower P-induced state-transition relation and $\Delta_T = \bigcup_{a_k \in \Sigma} \Delta_{T_{a_k}}$ is called the upper T-induced state-transition relation. The general fuzzy automaton $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ with state-transition relation Δ^{P_δ} is said to be the lower P_δ-induced general fuzzy automaton and we consider it as $\tilde{F}^{P_\delta}$ and the general fuzzy automaton $\tilde{F}$ with state-transition relation Δ_{T_δ} is said to be the upper T_δ-induced general fuzzy automaton and we consider it as $\tilde{F}_{T_\delta}$. We say that the general fuzzy automaton $\tilde{F}$ is recoverable from $T_\delta(P_\delta)$ if, for all $a_k \in \Sigma$, Δ is recoverable from $T_{\delta_{a_k}}(P_{\delta_{a_k}})$, i.e., if $\tilde{F} = \tilde{F}_{T_\delta}(\tilde{F} = \tilde{F}^{P_\delta})$.

Theorem 3.1. *Let* $\mathbf{M} = ([0,1]; \leq, 0, 1)$ *be a non-trivial complete lattice,*

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$$

be a general fuzzy automaton and $\mathbf{B}$ *be a bounded subposet of* $\mathbf{M}^Q$*. Let* $P_\delta : B \to (M^Q)^\Sigma$ *and* $T_\delta : B \to (M^Q)^\Sigma$ *be labeled transition functors constructed by means of* $\tilde{F}$*.*
Then for all $b_1, b_2 \in B$*,*
(i) $P_\delta(b_1) \leq b_2 \Longleftrightarrow b_1 \leq T_\delta(b_2)$*.*
Moreover, the following holds.
(ii) If $\Delta = \Delta_{T_\delta}$ *and* $T_\delta(B) \subseteq B^\Sigma$ *then* $\Delta = \Delta_{T_\delta} = \Delta^{P_\delta}$*.*
(iii) If $\Delta = \Delta^{P_\delta}$ *and* $P_\delta(B) \subseteq B^\Sigma$ *then* $\Delta = \Delta_{T_\delta} = \Delta^{P_\delta}$*.*

Proof. (i): Clearly for all $b_1, b_2 \in B$

$$\begin{aligned}
&P_\delta(b_1) \leq b_2 \Longleftrightarrow \forall a_k \in \Sigma, P_{\delta_{a_k}}(b_1) \leq b_2 \\
&\Longleftrightarrow \forall a_k \in \Sigma, q_i \in Q, P_{\delta_{a_k}}(b_1)(q_i) \leq b_2(q_i) \\
&\Leftrightarrow \forall a_k \in \Sigma, q_i \in Q, q_j \in Q_{pred}(q_i, a_k), b_1(q_j) \leq b_2(q_i) \\
&\Leftrightarrow \forall a_k \in \Sigma, q_j \in Q, q_i \in Q_{succ}(q_j, a_k), b_1(q_j) \leq b_2(q_i) \\
&\Leftrightarrow \forall a_k \in \Sigma, q_j \in Q, b_1(q_j) \leq T_{\delta_{a_k}}(b_2)(q_j) \\
&\Leftrightarrow \forall a_k \in \Sigma, b_1 \leq T_{\delta_{a_k}}(b_2) \\
&\Leftrightarrow b_1 \leq T_\delta(b_2).
\end{aligned}$$

(ii): Assume that $\Delta = \Delta_{T_\delta}$ and $T_\delta(B) \subseteq B^\Sigma$. We first show that $\Delta^{P_\delta} \subseteq \Delta_{T_\delta}$. Let $q_i, q_j \in Q$ and $(q_i, q_j) \in \Delta^{P_\delta}$. Let $b_1 \in B$. We put $b_2 = T_\delta(b_1)$. Partially (i) we have $P_\delta(T_\delta(b_1)) \le b_1$ and hence $T_\delta(b_1)(q_j) \le P_\delta(T_\delta(b_1))(q_i) \le b_1(q_i)$, i.e., $(q_i, q_j) \in \Delta_{T_\delta}$ and we have $\Delta^{P_\delta} \subseteq \Delta_{T_\delta}$. Then $\Delta \subseteq \Delta^{P_\delta} \subseteq \Delta_{T_\delta} = \Delta$ which yields the statement.

□

(iii): It follows from the same reasoning as in (ii).

Example 3.2. *Consider the general fuzzy automaton $\tilde{F}$ of Example 3.1. Let P be a restriction of the operator P_{δ_b} of Example 3.1 and Let T be a restriction of the operator T_{δ_b} of the same example. Let us compute Δ_T and Δ^P. We have $\Delta_T = \Delta^P = \{(q_0, q_1), (q_2, q_2)\}$. Hence the transition relation δ_b of Example 3.1 coincides with our induccd transition relations Δ_T and Δ^P. We can see from obove that the oprator T_{δ_b} bears the maximal amount of information about the transition relation δ_b on the subposet of $P_{\delta_b} o T_{\delta_b}$. The same conclusion holds for the operator P_{δ_b}.*

Example 3.3. *Consider the general fuzzy automaton $\tilde{F}$ of Example 3.1. Let us put $B = [0,1]^Q$. Let $P : [0,1]^Q \to [0,1]^Q$ and $T : [0,1]^Q \to [0,1]^Q$ be morphisms of partially ordered sets given as follows:*

$$\begin{array}{ll} T(0) = 0, & P(0) = 0, \\ T(0) = s_0, & P(s_0) = 0, \\ T(s_2) = s_1, & P(s_1) = s_2, \\ T(s_0') = s_2, & P(s_2) = s_0', \\ T(s_0') = s_0', & P(s_0') = s_0', \\ T(s_0') = s_1', & P(s_1') = s_0', \\ T(s_2) = s_2', & P(s_2') = s_2, \\ T(1) = 1, & P(1) = s_0'. \end{array}$$

Note that P coincides with the operator P_{δ_a} of Example 3.1, and T coincides with the operator T_{δ_a} of the same example. We have $\Delta_T = \Delta^P = \{(q_1, q_2), (q_2, q_1), (q_2, q_2)\}$. The transition relation δ_a of Example 3.1 coincides with or induces transition relation Δ_T and Δ^P.

The following corollary illustrates the situation in the case where our partially ordered set $\mathbf{B}$ of propositions is large enough, i.e., the case when $[0,1]^Q \subseteq B$.

Corollary 3.2. *Let $\mathbf{M}$ be a non-trivial complete lattice and $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ a general fuzzy automaton. Let $\mathbf{B}$ be a bounded subposet of $\mathbf{M}^Q$ so that $[0,1]^Q \subseteq B$. Then the general fuzzy automaton $\tilde{F}$ is recoverable both from P_δ and T_δ.*

Proof. Define, for all $q_i \in Q$, an element $b(q_i) \in [0,1]^Q \subseteq B$ by

$$[b(q_i)](q_j) = \begin{cases} 0 & if \quad q_i = q_j \\ \mu^{t_i}(q_j) & if \quad q_i \neq q_j \end{cases}.$$

Then $b(q_i) \in B$ satisfies the assumption of part (ii) of Theorem 3.1, i.e., $\Delta = \Delta_{T_\delta}$. Similarly, $\Delta = \Delta^{P_\delta}$.

□

In the following theorems, we are going to demonstrate that the state-transition relation on Q and the transition operators on $\mathbf{B}$ form a Galois connection. This is significant since in every Galois connection one of its components completely ascertains the second one and vice versa.

Let $\mathbf{M} = ([0,1]; \le, 0, 1)$ be a non-trivial complete lattice and Q a non-empty set of states of $\tilde{F}$. Let $\mathbf{B}$ be a bounded subposet of $\mathbf{M}^Q$, $(\xi(Q \times Q); \subseteq, \emptyset, Q \times Q)$ be the poset of all relations on Q and $(Map(\mathbf{B}, \mathbf{M}^Q); \sqsubseteq)$ be the poset of all order-preserving mappings $T : B \to (M^Q)^\Sigma$ so that $T(1) = 1$ and $T_1 \sqsubseteq T_2$ if and only if $T_2(b) \le T_1(b)$ for all $b \in B$. The smallest element of $(Map(\mathbf{B}, \mathbf{M}^Q); \sqsubseteq)$ is the constant mapping $\mathbf{1}$ so that $\mathbf{1}(b) = (1)_{q_i \in Q}$ for all $b \in B$. Let us put, for all $\Delta_T \in \xi(Q \times Q)$ and all $T \in Map(\mathbf{B}, \mathbf{M}^Q)$, $\phi(\Delta_T) = T_\delta$ and $\psi(T) = \Delta_{T_\delta}$.

Theorem 3.3. *Let $\mathbf{M} = ([0,1]; \le, 0, 1)$ be a non-trivial complete lattice and Q the set of states of general fuzzy automaton $\tilde{F}$ so that $\mathbf{B}$ is a bounded subposet of $\mathbf{M}^Q$. Then the couple (ϕ, ψ) is a Galois connection between $(\xi(Q \times Q); \subseteq, \emptyset, Q \times Q)$ and $(Map(\mathbf{B}, \mathbf{M}^Q); \sqsubseteq)$.*

Proof. It is clear that ϕ and ψ defined above are order-preserving mappings. It is enough to check that, for all $\Delta_T \in \xi(Q \times Q)$ and all $T \in Map(\mathbf{B}, \mathbf{M}^Q)$

$$\phi(\Delta_T) \sqsubseteq T \quad \text{if and only if} \quad \Delta_T \subseteq \psi(T).$$

Assume first that $\phi(\Delta_T) \sqsubseteq T$ holds and let $(q_i, q_j) \in \Delta_T$. Then, for all $b \in B$; we have $T(b)(q_i) \leq (\phi(\Delta_T)(b))(q_i) = T_\delta(b)(q_i) = \wedge\{b(q_j)|q_j \in Q_{succ}(q_i, a_k)\} \leq b(q_j)$. This yields that $(q_i, q_j) \in \psi(T) = \Delta_{T_\delta}$.
On the other hand, assume that $\Delta_T \subseteq \psi(T)$ and let $b \in B$, $q_i \in Q$. Either the set $\{q_j \in Q|(q_i, q_j) \in \Delta_T\} = \emptyset$ in which case $(T_\delta(b))(q_i) = 1$ which yields $(T(b))(q_i) \leq 1 = (\phi(\Delta_T)(b))(q_i)$ or $\{q_j \in Q|(q_i, q_j) \in \Delta_T\} \neq \emptyset$.
In the last case, we have $\{b(q_j) \in T|(q_i, q_j) \in \Delta_T\} \neq \emptyset$ and by the definition of $\phi(\Delta_T) = T_\delta$ we have $(T_\delta(b))(q_i) = \wedge\{b(q_j)|q_j \in Q_{succ}(q_i, a_k)\} \leq b(q_j)$ for all $q_j \in Q$ so that $(q_i, q_j) \in \Delta_T$. Since $\Delta_T \subseteq \psi(T)$ we have, for all $q_j \in Q$ so that $(q_i, q_j) \in \Delta_T$, that, for all $c \in B$, $(T(c))(q_i) \leq c(q_j)$. It follows that $(T(c))(q_i) \leq (T_\delta(c))(q_i) = (\phi(\Delta_T)(c))(q_i)$. But we have just proved that $\phi(\Delta_T) \sqsubseteq T$.

□

Remark 3.4. *We indicate that our recoverable relations from the respective upper transition operators are exactly fixpoints of the composition $\psi o \phi : \xi(Q \times Q) \to \xi(Q \times Q)$.*

Dually, let $\mathbf{M} = ([0,1]; \leq, 0, 1)$ be a non-trivial complete lattice and Q be the set of states of general fuzzy automata $\tilde{F}$. Let $\mathbf{A}$ be a bounded subposet of $\mathbf{M}^Q$, $(\xi(Q \times Q); \subseteq, \emptyset, Q \times Q)$ be the poset of all relations on Q and $(Map^0(\mathbf{A}, \mathbf{M}^Q); \leq)$ be the poset of all order-preserving mappings $P : A \to (M^Q)^\Sigma$ so that $P(0) = 0$ and $P_1 \leq P_2$ if and only if $P_1(a) \leq P_2(a)$ for all $a \in A$. The smallest element of $(Map^0(\mathbf{A}, \mathbf{M}^Q); \leq)$ is the constant 0 so that $\mathbf{0}(a) = (0)_{q_i \in Q}$ for all $a \in A$. Let us put, for all $\Delta_T \in \xi(Q \times Q)$ and all $P \in Map^0(\mathbf{A}, \mathbf{M}^Q)$, $\Phi(\Delta^P) = P_\delta$ and $\Psi(P) = \Delta^{P_\delta}$.

Theorem 3.5. *Let $\mathbf{M} = ([0,1]; \leq, 0, 1)$ be a non-trivial lattice and Q be the set of states of general fuzzy automata $\tilde{F}$ so that $\mathbf{A}$ is a bounded subposet of $\mathbf{M}^Q$. Then the couple (Φ, Ψ) is a Galois connection between $(\xi(Q \times Q); \subseteq, \emptyset, Q \times Q)$ and $(Map^0(\mathbf{A}, \mathbf{M}^Q); \leq)$.*

Proof. Consider $(Map^0(\mathbf{A}, \mathbf{M}^Q); \leq) = (Map(\mathbf{A}^{op}, (\mathbf{M}^{op})^Q); \sqsubseteq)$, $\Phi(\Delta^P) = P_\delta = \varphi(\Delta_{T^{-1}})$ and $\Psi(P) = \Delta^{P_\delta} = \psi(T^{-1})$. Then the proof is similar to that of Theorem 3.5.

□

Example 3.4. *Consider the general fuzzy automaton $\tilde{F}$, the set of propositions B and the state - transition relation Δ of Example 3.1. From Example 3.1, we know the labeled upper transition functor $T_\delta = (T_{\delta_a}, T_{\delta_b})$ and the labeled lower transition functor $P_\delta = (P_{\delta_a}, P_{\delta_b})$ from B to $([0,1]^Q)^\Sigma$. Since $B = [0,1]^Q$ we have $T_{\delta_a}(B) \cup T_{\delta_b}(B) \subseteq B$ and $P_{\delta_a}(B) \cup P_{\delta_b}(B) \subseteq B$. Now, we use T_δ for computing the transition relations, $\Delta_{T_{\delta_a}}$ and $\Delta_{T_{\delta_b}}$ (by the formula $(*)$ and Example 3.4) and P_δ for computing the transition relations $\Delta^{P_{\delta_a}}$ and $\Delta^{P_{\delta_b}}$ (by the formula $(**)$ and Example 3.4). We obtain by the Corollary 3.2 that $\Delta_{T_{\delta_a}} = \Delta^{P_{\delta_a}} = \delta_a$ and $\Delta_{T_{\delta_b}} = \Delta^{P_{\delta_b}} = \delta_b$. It follows that $\Delta_{T_\delta} = \Delta^{P_\delta} = \Delta_{T_{\delta_a}} \cup \Delta_{T_{\delta_b}} = \Delta$ i.e., Our given state-transition relation Δ simulatianeously is recoverable by the transition functors T_δ and P_δ. Hence these functors are carachteristics of the triple (B, Σ, Q).*

By a synthesis in theory of systems, it is usually meant that the task constructs a general fuzzy automaton $\tilde{F}$ which realizes a dynamic process at least partially known to the user.
Hence, we are given a description of this dynamic process and we know the set Σ of inputs. Our task is to set up the set Q of states and a relation Δ on Q labeled by elements from Σ so that the constructed general fuzzy automaton $\tilde{F}$ induces the logic, i.e., the partially ordered set of propositions which corresponds to the presented descriptions.
The algebraic tools which were collected in previous sections enable us to solve the mentioned task. In what follows, we represent a construction of Q and Δ which provides our logic with the transition functor representing the dynamics of our system. As mentioned in the previous section, our logic $\mathbf{B}$ will be considered to be a bounded subposet $\mathbf{B}$ of a power $\mathbf{M}^Q$ where $\mathbf{M}$ is a complete lattice of truth-values. Our logic $\mathbf{B}$ is equipped with a transition functor $T : B \to (M^Q)^\Sigma$ where Σ is a set of possible inputs. We ask that either $T = T_\delta$ or $T = P_\delta$.
Depending on the respective type of our submitted logic and of the properties of T we will introduce some possible solutions to this task.
For any bounded partially ordered set $\mathbf{B} = (B; \leq, 0, 1)$. We have a full set $S_\mathbf{B}$ of morphisms of bounded partially ordered set into the algebra regarded as a bounded partially ordered set $([0,1], \leq, 0, 1)$. The elements $h_D : B \to [0,1]$ of $S_\mathbf{B}$ (indexed by proper down-sets D of $\mathbf{B}$) are morphisms of bounded partially ordered sets defined by the prescription, for all $a_k \in \Sigma$

$$h_{D_{a_k}}(b) = \begin{cases} T_{a_k}(b) & if \quad b \notin D \\ 0 & if \quad b \in D \end{cases}.$$

In other words, every bounded partially ordered set $\mathbf{B}$ can be embedded into an algebra $[0,1]^S$ for a certain set S via the mapping $i^S_{\mathbf{B}}$.
Thus, it seems confident to apply the bounded partially ordered set $\mathbf{M} = ([0,1]; \leq, 0, 1)$ for the construction of our state-transition $\Delta \subseteq S_B \times \Sigma \times S_B$. As it was mentioned in the beginning of this section, we are interested in a construction of a general fuzzy automaton $\tilde{F}$ for a given set Σ of inputs and determined by a certain partially ordered set of propositions. We cannot assume that this set of propositions is necessarily a Boolean algebra. In the previous part, we supposed that this logic $\mathbf{B}$ is a bounded partially ordered set $\mathbf{B} = (B; \leq, 0, 1)$. Now, we are going to solve the situation when it is only a subset C of B.

Theorem 3.6. *Let $\mathbf{B} = (B; \leq, 0, 1)$ be a bounded partially ordered set so that $\mathbf{B}$ is a bounded subposet of M^{S_B}. Let $(C; \leq, 1)$ be a subposet of $\mathbf{B}$ including 1, and Σ a non-empty set. Let $T = (T_{a_k})_{a_k \in \Sigma}$ where $T_{a_k} : C \to M^{S_B}$ are morphisms of partially ordered sets so that $T_{a_K}(1) = 1$ for all $a_k \in \Sigma$. Let Δ_T be the labeled upper T-induced state-transition relation and $T_\delta : B \to (M^{S_B})^\Sigma$ be the labeled upper transition functor constructed by means of the upper T_δ-induced automaton $\tilde{F}_{T_\delta}$. Then, for all $b \in C$, $T(b) = T_\delta(b)$.*

Proof. Clearly, $T_\delta = (T_{\delta_{a_k}})_{a_k \in \Sigma}$ where $T_{\delta_{a_k}} : B \to M^{S_B}$ are morphisms of partially ordered sets for all $a_k \in \Sigma$. We write $\Delta_T = \bigcup_{a_k \in \Sigma} \Delta_{T_{a_k}}$ where $\Delta_{T_{a_k}}$, $a_k \in \Sigma$ are the T_{a_k}-induced relation by M. Let us choose $b \in C$ and $a_k \in \Sigma$ arbitrarily, but fixed. We have to check that $T_{a_k}(b) = T_{\delta_{a_k}}(b)$. Suppose that $h_D \in S_{\mathbf{B}}, a_k \in \Sigma$. It is enough to verify that $T_{a_k}(b)(h_D) = \wedge\{b(h_C) | h_C \in Q_{succ}(h_D, a_k)\}$. Evidently, for all $h_C \in S_{\mathbf{B}}$ so that $(h_D, h_C) \in \Delta_{T_{a_k}}$, $T_{a_k}(b)(h_D) \leq b(h_C)$. Hence $T_{a_k}(b)(h_D) \leq \wedge\{b(h_C) | h_C \in Q_{succ}(h_D, a_k)\}$. To get the other inequality, assume that

$$T_{a_k}(b)(h_D) < \wedge\{b(h_C) | h_C \in Q_{succ}(h_D, a_k)\}.$$

Then $T_{a_k}(b)(h_D) = 0$ and $\wedge\{b(h_C) | h_C \in Q_{succ}(h_D,\, a_k)\} \neq 0$.
Put $V_{a_k} = \{z \in B | \exists y \in C, T_{a_k}(y)(h_D) \neq 0 \text{ and } y \leq z\}$. It follows that $b \notin V_{a_k}$ and V_{a_k} is an upper set of $\mathbf{B}$ so that $1 \in V_{a_k}$ (since $h_{V_{a_k}}(1) = T_{a_k}(1) = 1 \neq 0$). Let W_{a_k} be a maximal proper upper set of $\mathbf{B}$ including V_{a_k} so that $b \notin W_{a_k}$. Put $U_{a_k} = B \backslash W_{a_k}$. Then U_{a_k} is a proper down-set, $0 \in U_{a_k}$, $h_{U_{a_k}}(b) = 0$ and $h_{U_{a_k}}(z) \neq 0$ for all $z \in V_{a_k}$, i.e., $h_{U_{a_k}} \in S_{\mathbf{B}}$ so that $T_{a_k}(a)(h_D) \leq a(h_{U_{a_k}})$ for all $a \in C$. But this yields $(h_D, h_{U_{a_k}}) \in \Delta_{T_{a_k}}$, i.e., $0 \neq \wedge\{b(h_C) | h_C \in Q_{succ}(h_D, a_k)\} \leq b(h_{U_{a_k}}) = h_{U_{a_k}}(b) = 0$, a contradiction.

□

Using the relation Δ^P instead of Δ_T, we can obtain a statement dual of Theorem 3.6.
Consequently, with respect to the above mentioned materials and Theorem 3.6, we obtain the the upper T_δ-induced general fuzzy automaton $\tilde{F}_{T_\delta}$ as ten-tuple machine denoted with $\tilde{F}_{T_\delta} = (S_B, \Sigma, \tilde{R} = \{(h_{\{0\}}, \mu^{t_0}(h_{\{0\}}))\}, Z, \omega, \delta, \tilde{\delta}, T_\delta, F_1, F_2)$ where,
(i) S_B is the set of states, $S_B = \{h_D : B \to [0,1], D \subseteq B\}$ so that for all $a_k \in \Sigma$

$$h_{D_{a_k}}(b) = \begin{cases} T_{\delta_{a_k}}(b) & if \quad b \notin D \\ 0 & if \quad b \in D \end{cases},$$

(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \dots, a_m\}$,
(iii) $\tilde{R} = \{(h_{\{0\}}, \mu^{t_0}(h_{\{0\}}))\}$ is the set of fuzzy start state,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \dots, b_k\}$,
(v) $\omega : S_B \to Z$ is the output function,
(vi) $\delta : S_B \times \Sigma \times S_B \to [0,1]$ is the transition function defined by:

$$\delta(h_D, a_k, h_C) = h_{D_{a_k}}(b) \vee h_{C_{a_k}}(b)$$

for all $b \in B$ and $D, C \subseteq B$,
(vii) $\tilde{\delta} : (S_B \times [0,1]) \times \Sigma \times S_B \to [0,1]$ is the augmented transition function so that:
$\mu^{t+1}(h_D) = \tilde{\delta}((h_D, \mu^t(h_D)), a_k, h_C) = F_1(\mu^t(h_D), \delta(h_D, a_k, h_C))$,
(viii) $T_\delta : B \to (M^{S_B})^\Sigma$ is the labeled upper transition functor so that $T_{\delta_{a_k}}(b)(h_D) = \wedge\{b(h_C) | h_C \in Q_{succ}(h_D, a_k)\}$ for all $a_k \in \Sigma$,
(ix) $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function,
(x) $F_2 : [0,1]^\star \to [0,1]$ is called multi-membership resolution function.

Example 3.5. *Consider again the set $Q = \{q_0, q_1, q_2\}$ of states, the set $\Sigma = \{a, b\}$, and the set of propositions $B = [0,1]^Q$ of Example 3.1. Assume that $C = \{0, s_2, s'_0, s'_1, 1\} \subseteq B$ from the logic B of Example 3.1. Assume further that our partially known transition operator T from C to $([0,1]^Q)^\Sigma$ is given as follows:*

$$T_a(0) = 0, \qquad T_a(s'_0) = s'_0, \qquad T_b(1) = 1,$$

$$T_a(s_0') = s_2, \qquad T_a(s_0') = s_1', \qquad T_b(s_2) = s_0',$$
$$T_a(1) = 1, \qquad T_b(s_2) = s_2, \qquad T_b(s_0') = s_1'.$$

Note that we have chosen T as a resteriction of the operator T_δ from Example 3.1 on the set C. Then, by an easy computation we optain from $(*)$ that $\Delta_T = \Delta_{T_a} \cup \Delta_{T_b}$ where $\Delta_{T_a} = \{(q_1, q_2), (q_2, q_1), (q_2, q_2)\}$ and $\Delta_{T_b} = \{(q_0, q_1), (q_2, q_2)\}$. From Theorem 3.6, we have that T is a restriction of the operator T_{Δ_T} on the set C. Moreover, we can see that our state transition relation Δ from Example 3.1 coincids with the induced state-transition relation Δ_T. i.e., our partialy known transition operator T has given us a full information about the general fuzzy automaton $\tilde{F}$ from Example 3.1.

4. Conclusion

By a general fuzzy automaton, we contributed to show a set of propositions related to a given automaton and that the truth-values are depended on the states, inputs and membership values of active states at time t. This approach enables us to consider automata from a different point of view which is more close to logical treatment and helps us make estimations about the behavior of automaton particularly in a nondeterministic mode. The logic consists of propositions on the given GFA and its dynamic nature is stated by means of the so-called transition functor. This logic enables us to derive a certain relation on the set of states labeled by inputs. In fact, we showed that if our set of propositions is large enough, this recovering of the transition relation is possible. Moreover, a very challenging implication of our approach is that a zero-weight transition is possible and is different from no transition. A zero-weight transition may give rise to the activation of a successor due to the activation of its predecessor. While in all types of conventional automata, a zero-weight transition means no transition, in our approach to general fuzzy automata a zero-weight transition does not necessarily imply no transition. That is why we use [0; 1] as the fuzzy interval. Then, in this paper we studied the theory of general fuzzy automata by using the concepts of operators. Such operators help us in the algebraic study of general fuzzy automata theory and provide a platform to use fuzzy topological therein in the future.

Compliance with Financial and Ethical Standards
Funding: This study was not funded by any organization or institute.
Conflict of Interest: The authors declare that they have no conflict of interest.
Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] K. Abolpour, M. M. Zahedi, M. Shamsizadeh, BL-general fuzzy automata and minimal realization: based on the associated categories, Iranian Journal of Fuzzy Systems, 17 (2020) 155-169.

[2] K. Abolpour, M. M. Zahedi, BL-general fuzzy automata and accept behavior, Journal of Applied Mathematics and Computations, 38 (2012) 103-118.

[3] K. Abolpour, M. M. Zahedi, Isomorphism between two BL-general fuzzy automata, Soft Computing, 16 (2012) 729-736.

[4] K. Abolpour, M. M. Zahedi, General fuzzy automata based on complete residuated lattice-valued, Iranian Journal of Fuzzy Systems, 14 (2017) 103-121.

[5] Z. Bavel, J. W. Thomas, On the decomposability of monadic algebras and automata, In: Proceedings of the 8th Annual Symposium on Switching and Automata Theory, (1967) 322-335.

[6] A. Bisio, G. M. Dariano, P. Perinotti, A. Tosini, Free quantum field theory from quantum cellular automata, Foundations of Physics, 45 (2015) 1137-1152.

[7] I.Chajada, J. Paseka, Transition operators assigned to physical systems, Reports on Mathematical Physics, 78 (2016) 259-280.

[8] P. Das, A fuzzy topology associated with a fuzzy finite state machine, Fuzzy Sets and Systems, 105 (1999) 469-479.

[9] M. Doostfatemeh, S. C. Kremer, New directions in fuzzy automata, International Journal of Approximate Reasoning, 38 (2005) 175-214.

[10] W. M. L. Holcombe, Algebraic Automata Theory, Cambridge University Press, Cambridge 1982.

[11] M. Horry, M. M. Zahedi, Some (fuzzy) topologies on general fuzzy automata, Iranian Journal of Fuzzy Systems, 10 (2013), 73-89.

[12] M. Ito, Algebraic structures of automata, Theoretical Computer Science, 429 (2012) 164-168.

[13] J. H. Jin, Q. G. Li, Y. M. Li, Algebraic properties of L-fuzzy finite automata, Information Sciences, 234 (2013) 182-202.

[14] Y. H. Kim, J. G. Kim, S. J. Cho, Products of T-generalized state machines and T-generalized transformation semigroups, Fuzzy Sets and Systems, 93 (1998) 87-97.

[15] F. C. Liu, D. Qiu, Diagnosability of fuzzy discrete event systems: f Fuzzy approach, IEEE Transactions on Fuzzy Systems, 17 (2009) 372-384.

[16] D. S. Malik, J. N. Mordeson, M. K. Sen, Submachines of fuzzy finite state machine, The Journal of Fuzzy Mathematics, 2 (1994) 781-792.

[17] J. Mockor, A category of fuzzy automata, International Journal of General Systems, 20 (1991) 73-82.

[18] J. Mockor, Fuzzy and non-deterministic automata, Soft Computing, 3 (1999) 221-226.

[19] J. Mockor, Semigroup homomorphisms and fuzzy automata, Soft Computing, 6 (2002) 423-427.

[20] P. Pal, S. P. Tiwari, R. Verma, On different operators in automata theory based on residuated and co-residuated lattices, New Mathematics and Natural Computations, 15 (2019) 169-190.

[21] K. Peeva, Behavior, reduction and minimization of finite L-automata, Fuzzy Sets and Systems, 28 (1988) 171-181.

[22] K. Peeva, Fuzzy acceptors for syntactic pattern recognition, International Journal of Approximate Reasoning, 5 (1991) 291-306.

[23] D. Qiu, Automata theory based on complete residuated lattice-valued logic (I), Science in China Series F Information Sciences, 44 (2001) 419-429.

[24] D. Qiu, Automata theory based on complete residuated lattice-valued logic (II), Science in China Series F Information Sciences, 45 (2002) 442-452.

[25] D. Qiu, Automata theory based on quantum logic: some characterizations, Information and Computation, 190 (2004) 179-195.

[26] D. Qiu, Characterizations of fuzzy finite automata, Fuzzy Sets and Systems, 141 (2004) 391-414.

[27] E. S. Santos, Maximin automata, Information and Control, 12 (1968), 367-377.

[28] A. K. Srivastava, S. P. Tiwari, A topology for fuzzy automata, Lecture Notes in Artificial Intelligence, 2275 (2002) 485-490.

[29] S. P. Tiwari, V. Gautam B. Davvaz, On minimal realization for a fuzzy language and Brzozowskis algorithm, Journal of Intelligent & Fuzzy Systems, 29 (2015) 1949-1956.

[30] S. P. Tiwari, S. Sharan, Fuzzy automata based on lattice-ordered monoids with algebraic and topological aspects, Fuzzy Information and Engineering, 4 (2012) 155-164.

[31] S. P. Tiwari, A. K. Singh, S. Sharan, V. K. Yadav, Bifuzzy core of fuzzy automata, Iranian Journal of Fuzzy Systems, 12 (2015) 63-73.

[32] S. P. Tiwari, A. K. Srivastava, On a decomposition of fuzzy automata, Fuzzy Sets and Systems, 151 (2005) 503-511.

[33] S. P. Tiwari, V. K. Yadav, A. K. Singh, Construction of a minimal realization and monoid for a fuzzy language, a categorical approach, Journal of Applied Mathematics and Computing, 47 (2015) 401-416.

[34] V. Topencharov, K. Peeva, Equivalence, reduction and minimization of finite fuzzy automata, Journal of Mathematical Analysis and applications, 84 (1981) 270-281.

[35] W. G. Wee, On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classification, Ph.D. Thesis, Purdue University, Lafayette, IN, 1967.

[36] L. A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.

Please cite this article using:

Khadijeh Abolpour, Mohammad Mehdi Zahedi, Marzieh Shamsizadeh, New directions in general fuzzy automata: a dynamic-logical view, AUT J. Math. Com., 1(2) (2020) 251-262
DOI: 10.22060/ajmc.2020.18629.1040