New directions in general fuzzy automata: a dynamic-logical view

Khadijeh Abolpour\textsuperscript{*a}, Mohammad Mehdi Zahedi\textsuperscript{b,c}, Marzieh Shamsizadeh\textsuperscript{b,c}

\textsuperscript{a}Dept. of Math., Shiraz Branch, Islamic Azad University, Shiraz, Iran
\textsuperscript{b}Dept. of Math., Graduate University of Advanced Technology, Kerman, Iran
\textsuperscript{c}Dept. of Math., Behbahan Khatam Alيانia University of Technology, Behbahan, Iran

ABSTRACT: In the current study, by a general fuzzy automaton we aim at showing a set of propositions related to a given automaton showing that the truth-values are depended on the states, inputs and membership values of active states at time t. This new approach enables us to consider automata from a different point of view which is more close to logical treatment and helps us make estimations about the behavior of automaton particularly in a nondeterministic mode. The logic consists of propositions on the given GFA and its dynamic nature is stated by means of the so-called transition functor. This logic enables us to derive a certain relation on the set of states labeled by inputs. In fact, it is shown that if our set of propositions is large enough, this recovering of the transition relation is possible. Through a synthesis in the theory of systems, this study contributes to construct a general fuzzy automaton which realizes a dynamic process at least partially known to the user, which has been fully achieved in Theorem 3.6. Also, we study the theory of general fuzzy automata by using the concepts of operators. Such operators help us in the algebraic study of general fuzzy automata theory and provide a platform to use fuzzy topological therein. Further, a Galois connection is obtained between the state-transition relation on states and the transition operators on propositions. To illustrate the proposed approach, the subject matter is more elaborated in detail through examples.

1. Introduction

The concept of a finite automaton is well-known. Algebraic study of automata has been carried out by many authors in many forms (cf., e.g., [5, 10, 12]). After the introduction of fuzzy set in 1965 by Zadeh [36], a number of concepts in mathematics and other areas were fuzzified. Among the first such concepts was the concept of fuzzy automaton firstly proposed by Wee [35] and Santos [27], to deal with the notions such as vagueness and imprecision, frequently encountered in the study of natural languages. Further, Malik et. al. [16] introduced a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) and contributed greatly towards the algebraic study of fuzzy automata and fuzzy languages. M. Doostfatemeh and S. C. Kremer [9] used an extension of the notion of fuzzy automata and gave the concept of general fuzzy automata (for simplicity GFA). Their key motivation of introducing the notion general fuzzy automata was the insufficiency of the current literature to handle the applications which rely on fuzzy automata as a modeling tool, and assign membership values to active states of a fuzzy automaton. It will be interesting to see how the developed concepts and algorithms can be used in practice. A very interesting and challenging implication of our approach is that a zero-weight transition is possible and is different from no transition. A zero-weight transition may give rise to the activation of a successor due to the activation of its predecessor. A number of researchers have contributed to the growth of fuzzy automata theory.

*Corresponding author.
E-mail addresses: Abolpor_kh@yahoo.com, zahedi_mm@kgut.ac.ir, shamsizadeh.m@gmail.com

251
Among these works, the work of Das [8] is towards the fuzzy topological characterization of a fuzzy automaton; the work of Jin and his coworkers [13] is towards the algebraic study of fuzzy automata based on po-monoids; the work of Kim, Kim and Cho [14] is towards the algebraic study of fuzzy automata theory; the work of Mockor [17, 18, 19] is towards the use of categorical concepts in the study of fuzzy automata theory; the work of Abolpour and Zahedi [1, 2, 3, 4] is towards the use of categorical concepts in the study of general fuzzy automata with membership values in different lattice structures; the work of Horro and Zahedi [11] is towards the use of fuzzy topologies for the study of a max-min general fuzzy automaton; the work of Qiu [23, 24, 25, 26] is towards the algebraic, topological and categorical study of fuzzy automata theory based on residuated lattices; the work of Peeva [21, 22] is on the study of minimizing the states of fuzzy automata and its application to study pattern recognition; the work of Pal and the coworkers [20] is towards the study of fuzzy automaton based on residuated and co-residuated lattice. In previous studies on dynamic logic, all considerations are made on a physical system which is characterized by its states and transitions among the states. In fact, a dynamic logic [9] (Active state set) Knowing that the entered input prior for time \( t \) has been \( a_k \), active states at time \( t \) are those states to which there is at least one transition on the input symbol \( a_k \). Then, the fuzzy set of all active states at \( t \) (ordered pairs of states and their \( m_d \)'s) is called active state set at time \( t \), and is denoted as \( Q_{\text{act}}(t) \).

**2. preliminaries**

In this section, the basic definitions and theorems used for the concepts in the next parts will be presented in detail.

**Definition 2.1.** [9] A fuzzy set \( \mu_Q \) defined on a set \( Q \) (discrete or continuous), is a function mapping each element of \( Q \) to a unique element of the interval \([0,1]\).

\[
\mu_Q : Q \rightarrow [0,1]
\]

Then, the fuzzy power set of \( Q \) denoted as \( \hat{P}(Q) \), is the set of all fuzzy subsets \( \mu_Q \), which can be defined on the set \( Q \).

\[
\hat{P}(Q) = \{ \mu_Q | \mu_Q : Q \rightarrow [0,1] \}
\]

**Definition 2.2.** [9] (Active state set) Knowing that the entered input prior for time \( t \) has been \( a_k \), active states at time \( t \) are those states to which there is at least one transition on the input symbol \( a_k \). Then, the fuzzy set of all active states at \( t \) (ordered pairs of states and their \( m_d \)'s) is called active state set at time \( t \), and is denoted as \( Q_{\text{act}}(t) \).

**Definition 2.3.** [9] A general fuzzy automaton (GFA) is considered as

\[
\hat{F} = (Q, \Sigma, \hat{R}, Z, \delta, \omega, F_1, F_2),
\]

where (i) \( Q \) is a finite set of states, \( Q = \{ q_1, q_2, \ldots, q_n \} \), (ii) \( \Sigma \) is a finite set of input symbols, \( \Sigma = \{ a_1, a_2, \ldots, a_m \} \), (iii) \( \hat{R} \) is the set of fuzzy start states, \( \hat{R} \subseteq \hat{P}(Q) \), (iv) \( Z \) is a finite set of output symbols, \( Z = \{ b_1, b_2, \ldots, b_k \} \),
(v) $\omega : Q \to Z$ is the output function, (vi) $\bar{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function. (vii) Function $\bar{F}_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function. Function $\bar{F}_1(\mu, \delta)$, as is seen, is motivated by two parameters $\mu$ and $\delta$, where $\mu$ is the membership value of a predecessor and $\delta$ is the weight of a transition.

With this definition, the process that occurs upon the transition from state $q_i$ to $q_j$ an input $a_k$ is characterized by:

$$\mu^{t+1}(q_j) = \bar{\delta}(q_i, \mu^t(q_i), a_k, q_j) = \bar{F}_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

It means that membership value (mv) of the state $q_i$ at time $t+1$ is calculated by function $\bar{F}_1$ utilizing both the membership value of $q_i$ at time $t$ and the weight of the transition.

There have been many options for the function $\bar{F}_1(\mu, \delta)$. For instance, it can be $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$, $\frac{\mu + \delta}{2}$, or any other pertinent mathematical functions.

As it can be observed in the above mentioned, associated with each fuzzy transition, there exists a membership value (mv) in unit interval $[0,1]$. We identify this membership value as the weight of the transition. The transition from state $q_i$ (current state) to state $q_j$ (next state) upon input $a_k$ is designated as $\delta(q_i, a_k, q_j)$. Hereafter, we apply this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition; otherwise, it identifies the transition itself. The set of all transitions of a general fuzzy automaton $\bar{F}$, is denoted as $\Delta_{\bar{F}}$. However, whenever it is understood we remove the subscript, and write simply $\Delta$.

Concerning this, we say that $\Delta$ is a state-transition relation and it is regarded as a dynamics of $\bar{F}$. On the other hand, we regularly formulate certain propositions on an automaton $\bar{F}$ and draw conclusions from the behavior of $\bar{F}$ in the present or in the future.

(viii) $F_2 : [0,1]^* \to [0,1]$, is called multi-membership resolution function. The multi-membership resolution function determines the multi-membership active states and allocates a single membership value to them.

We let $Q_{act}(t_i)$ be the set of all active states at time $t_i$, $\forall_1 t_i \geq 0$. We have $Q_{act}(t_0) = \bar{R}$ and $Q_{act}(t_i) = \{(q, \mu^t(q_i))| q' \in Q_{act}(t_{i-1}), \exists a, q, q' \in Q, \delta(q', a, q) \in \Delta, \forall_1 t_i \geq 1$. Since $Q_{act}(t_i)$ is a fuzzy set, to demonstrate that a state $q$ belongs to $Q_{act}(t_i)$ and $T$ is a subset of $Q_{act}(t_i)$, we should write: $q \in \text{Domain}(Q_{act}(t_i))$ and $T \subseteq \text{Domain}(Q_{act}(t_i))$; henceforth. We simply specify them by: $q \in Q_{act}(t_i)$ and $T \subseteq Q_{act}(t_i)$.

**Definition 2.4.** [7] Let $S$ be a non-empty set. Every subset $R \subseteq S \times S$ is called a relation on $S$ and we declare that the couple $(S, R)$ is a transition frame.

**Definition 2.5.** [7] A mapping $f$ is called order - preserving or monotone if $a, b \in A$ and $a \leq b$ together imply $f(a) \leq f(b)$ and order-reflecting mapping $f$ if $a, b \in A$ and $f(a) \leq f(b)$ together imply $a \leq b$. A bijective order-preserving and order-reflecting mapping $f : A \to B$ is called an isomorphism and then we state that the partially ordered sets $(A; \leq)$ and $(B; \leq)$ are isomorphic.

**Definition 2.6.** [7] Let $(A; \leq)$ and $(B; \leq)$ be partially ordered sets. A mapping $f : A \to B$ is called residuated if there is a mapping $g : B \to A$ so that $f(a) \leq b$ if and only if $a \leq g(b)$ for all $a \in A$ and $b \in B$. In this case, we state that $f$ and $g$ form a residuated pair or that the pair $(f, g)$ is a (monotone) Galois connection. The role of Galois connections is indispensable for our constructions.

**Definition 2.7.** [7] If a partially ordered set $A$ has both a bottom and a top element, it will be called bounded; the pertinent notation for a bounded partially ordered set is $(A; \leq, 0, 1)$. Let $(A; \leq, 0, 1)$ and $(B; \leq, 0, 1)$ be bounded partially ordered sets. A morphism $f : A \to B$ of bounded partially ordered sets is an order, top element and bottom element preserving map.

**Observation 2.8.** [7] Let $A$ and $M$ be bounded partially ordered sets, and $h_s : A \to M$, $s \in S$, morphisms of bounded partially ordered sets. The conditions are equivalent:

(i) $\forall s \in S$ $h_s(a) \leq h_s(b) \Rightarrow a \leq b$ for any element $a, b \in A$;

(ii) The map $i^S_s : A \to M^S$ defined by $i^S_s(a) = (h_s(a))_{s \in S}$ for all $a \in A$ is order-reflecting.

Then, we declare that $\{h_s : A \to M; s \in S\}$ is a full set of order-preserving mappings concerning $M$. Note that in this situation we may specify $A$ with a bounded subposets of $M^S$ because $i^S_s$ is an order reflecting morphism alias embedding of bounded partially ordered sets. For any $s \in S$ and any $p = (p_s)_{s \in S}$ we indicate by $p(s)$ the $s$-th projection $p_s$. Note that $i^S_s(a)(s) = h_s(a)$ for all $a \in A$ and all $s \in S$.

**Definition 2.9.** [7] Let $A = (A; \leq, 0, 1)$ and $B = (B; \leq, 0, 1)$ be bounded posets with a full set $S$ of morphisms of bounded posets into a non-trivial complete lattice $M$. We may assume that $A$ and $B$ are bounded subposets of $M^S$.

Let $P : A \to B$ and $T : B \to A$ be morphisms of posets. Let us define the relations

$$R_T = \{(s, t) \in S \times S | (\forall b \in B)(s(T(b)) \leq t(b))\}$$

and

$$R^P = \{(s, t) \in S \times S | (\forall a \in A)(s(a) \leq t(P(a)))\}.$$
Lemma 2.10. [7] Let $M$ be a non-trivial complete lattice and $S$ a non-empty set so that $A$ and $B$ are bounded subposets of $M^S$. Let $P : A \rightarrow M^S$ and $T : B \rightarrow M^S$ be morphisms of posets so that, for all $a \in A$ and all $b \in B$,

$$P(a) \leq b \Leftrightarrow a \leq T(b).$$

(a) If $P(A) \subseteq B$ then $R_T \subseteq R_P$.
(b) If $T(B) \subseteq A$ then $R_P \subseteq R_T$.
(c) If $P(A) \subseteq B$ and $T(B) \subseteq A$ then $R_T = R_P$.

3. Construction of general fuzzy automata by dynamic logical view

One of the most important indicators in general fuzzy automaton is the existence of active state membership values based on which we obtain different results. Therefore, we use active state membership values to construct the logic structure. As a result, both active state membership values and the logical structure created by them are discussed in this article. This section aims at deriving the logic $B$ which is a set of propositions about the general fuzzy automaton $\tilde{F}$ formulated by the observer and constructing an ordered algebra structure on $B$. If we fix an input $a_k \in \Sigma$ at time $t_i$ the proposition $\alpha|_{a_k}$ can be computed by $\mu^i(q_i)$ if the general fuzzy automaton $\tilde{F}$ is in the state $q_i$ at time $t_i$ otherwise $\alpha|_{a_k}$ is 0 if $\tilde{F}$ is not in the active state $q_i$. Thus, for each state $q_i \in Q$ we can assess the truth value of $\alpha|_{a_k}$, it is noted by $\alpha|_{a_k}(q_i)$. As explained above, $\alpha|_{a_k}(q_i) \in [0, 1]$. We can establish the order $\leq$ on $B$ as follows: for $a, b, \in B, a \leq b$ if and only if $a(q_i) \leq b(q_i)$ for all $q_i \in Q$. One can instantly check that the contradiction, i.e., the proposition with constant truth value 0, is the least element and the tautology, i.e., the proposition with the constant truth value 1 is the greatest component of the partially order set ($B; \leq$). Note that any component $\alpha|_{a_k}$ is the maximum membership values of active states at time $t_i$, for any $i \geq 0$. This fact will be stated by the notation $B = (B; \leq, 0, 1)$ for the bounded partially ordered set of proposition about the general fuzzy automaton $\tilde{F}$. Every automaton $\tilde{F}$ will be identified with the triple $(B, \Sigma, Q)$, where $B$ is the set of propositions about $\tilde{F}$, $\Sigma$ is the set of possible inputs and $Q$ is the set of states on $\tilde{F}$. In what follows, the truth-values of our logic $B$ will be considered to be from the complete lattice $M = ([0, 1]; \leq, 0, 1)$. Thus $B$ will be bounded subposet of $M^Q$ for the complete lattice $M$ of truth-values.

We are given a set of labeled transitions $\Delta \subseteq Q \times \Sigma \times Q$ so that for an input $a_k \in \Sigma$, $\tilde{F}$ can go from $q_i$ to $q_j$ provided $\delta(q_i, a_k, q_j) \in \Delta$. As in the following, let $M = ([0, 1]; \leq, 0, 1)$ be a bounded partially ordered set and the bounded subposets $A = (A; \leq, 1)$ and $B = (B; \leq, 0, 1)$ of $M^Q$ will stand for the possibly different logics of propositions pertaining to our automaton $\tilde{F}$, a corresponding set of states $Q$, and a state-transition relation $\Delta$ on $Q$. The operator $T_3 : B \rightarrow (M^Q)\Sigma$ will prescribe to a proposition $b \in B$ about $\tilde{F}$ a new proposition $T_3(b) \in (M^Q)^\Sigma$ so that the truth value of $T_3(b)$ in state $q_m \in Q$ is the greatest truth value that is smaller than or equal to the corresponding truth values of $b$ in all states of $Q_{\text{succ}}(q_m, a_k)$. If there exists no such state, the truth value of $T_3(b)$ in state $q_m$ will be 1. Similarly, the operator $P_3 : A \rightarrow (M^Q)\Sigma$ will prescribe to a proposition $a \in A$ about $\tilde{F}$ a new proposition $P_3(a) \in (M^Q)\Sigma$ so that the truth value of $P_3(a)$ in state $q_m \in Q$ is the smallest truth value that is greater than or equivalent to the corresponding truth values of $a$ in all states of $Q_{\text{pred}}(q_m, a_k)$. If there is no such state, the truth value of $P_3(a)$ in state $q_m$ will be 0. Reflect on a complete lattice $M = ([0, 1]; \leq, 0, 1)$ and let $A = (A; \leq, 0, 1)$ and $B = (B; \leq, 0, 1)$ be bounded partially ordered sets with a full set $Q$ of morphisms of bounded partially ordered sets into a non-trivial complete lattice $M$. We may assume that $A$ and $B$ are bounded subposets of $M^Q$. Additionally, let $(Q, \Delta)$ be a transition frame. We can define mappings $P_5 : A \rightarrow (M^Q)^\Sigma$ and $T_5 : B \rightarrow (M^Q)^\Sigma$ as follows:

For all $b \in B$ and $q_m \in Q, a_k \in \Sigma$,

$$T_{a_k}(b)(q_m) = \land_M \{b(q_j) | q_j \in Q_{\text{succ}}(q_m, a_k)\}, \quad (*)$$

where

$$Q_{\text{succ}}(q_m, a_k) = \{q_j|\delta(q_m, a_k, q_j) \in \Delta\},$$

and for all $a \in A$

$$P_{a_k}(a)(q_m) = \lor_M \{a(q_j) | q_j \in Q_{\text{pred}}(q_m, a_k)\}, \quad (***)$$

where

$$Q_{\text{pred}}(q_m, a_k) = \{q_j|\delta(q_j, a_k, q_m) \in \Delta\}.$$
Example 3.1. Consider the GFA in Figure 1. It is specified as: $\tilde{F} = (Q, \Sigma, R, Z, \omega, \delta, F_1, F_2)$, where $Q = \{q_0, q_1, q_2\}$ is the set of states, $\Sigma = \{a, b\}$ is the set of input symbols, $R = \{(q_0, 1)\}$, $Z = \emptyset$ and $\omega$ is not applicable. If we choose $F_1(\mu, \delta) = \delta, F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^i(q_i), \delta(q_i, q_k, q_m)))$, then we have:

![Figure 1: The GFA of Example 3.1](image)

The set $B = \{0, s_0, s_1, s_2, s_0', s_1', s_2', 0\}$ of possible propositions $B$ about the automaton $\tilde{F}$ is as follows:

Table 1: Active states and their membership values (mv) at different times in Example 3.1 upon input string "$ba^2b$"

<table>
<thead>
<tr>
<th>time</th>
<th>$t_0$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>input</td>
<td>$\land$</td>
<td>$b$</td>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>$Q_{act}(t_i)$</td>
<td>$q_0$</td>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_1$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>mv</td>
<td>1</td>
<td>0.4</td>
<td>0.3</td>
<td>0.8</td>
<td>0.1</td>
</tr>
</tbody>
</table>

- 0 means that the GFA is not in active states of $Q$, 
- $s_0$ means that the GFA is in active state $q_0$, 
- $s_1$ means that the GFA is in active state $q_1$, 
- $s_2$ means that the GFA is in active state $q_2$, 
- $s_0'$ means that the GFA is either in active state $q_1$ or in the active state $q_2$, 
- $s_1'$ means that the GFA is either in active state $q_0$ or in the active state $q_2$, 
- $s_2'$ means that the GFA is either in active state $q_0$ or in the active state $q_1$, 
- 1 means that the GFA is in at least one active state of $Q$.

We may have $B$ with the algebra $[0, 1]^Q$ as follows: $0 = (0, 0, 0), s_0 = (1, 0, 0), s_1 = (0, 0.8, 0), s_2 = (0, 0.35, 0), s_0' = (0, 0.8, 0.35), s_1' = (1, 0.35, 0), s_2' = (1, 0.8, 0), 1 = (1, 0.8, 0.35).

Here, $\alpha(q_i)$ is the maximum membership values of active states at time $t_i$ for any $i \geq 0$. We have $\delta_0 = \{(q_1, q_2), (q_2, q_1), (q_2, q_2)\}$ and $\delta_0^* = \{(q_0, q_1), (q_1, q_2), (q_2, q_1), (q_2, q_2)\}$. Using our formulas ($*$) and (**), we can obtain the upper transition functors $T_{\alpha}, T_{\delta_0}: B \rightarrow [0, 1]^Q$ and the lower transition functors $P_{\alpha}, P_{\delta_0}: B \rightarrow [0, 1]^Q$ as follows:

- $T_{\delta_0}(0) = 0, T_{\alpha}(0) = 0$,
- $T_{\delta_0}(0) = s_0, T_{\alpha}(s_0) = 0$,
- $T_{\delta_0}(s_0) = s_1, T_{\alpha}(s_1) = s_2$,
- $T_{\delta_0}(s_1) = s_2, T_{\alpha}(s_2) = s_0$,
- $T_{\delta_0}(s_2) = s_0', T_{\alpha}(s_0') = s_0$,
- $T_{\delta_0}(s_0') = s_1', T_{\alpha}(s_1') = s_0$,
- $T_{\delta_0}(s_1') = s_2', T_{\alpha}(s_2') = s_2$,
- $T_{\delta_0}(s_2') = s_2, T_{\alpha}(1) = 1$,
- $P_{\alpha}(1) = 1$.
E.g. $T_{b_1}(s_2) = s_1$ meaning that if the GFA is in active state $q_1$, when entering input $a$, will change to $q_2$ and $P_{b_1}(s_2) = s'_1$ meaning that if the GFA is in active state $q_2$, when entering $a$, will change to $q_1$ or $q_2$.

Let us define the relations

$$
\Delta_{T_b} = \{(q_i, q_j) \in Q \times Q | \text{for all } b \in B, T_b(b)(q_i) \leq b(q_j)\},
$$

$$
\Delta^P = \{(q_i, q_j) \in Q \times Q | \text{for all } a, a(q_i) \leq P_b(a)(q_j)\}.
$$

In this situation, we assert that $\Delta$ is recoverable from $T_B$ or that $\Delta$ is recoverable from $P_B$. We claim that $\Delta$ is recoverable if it is recoverable both from $T_B$ and $P_B$.

Let us consider a general fuzzy automaton $\bar{F} = (Q, \Sigma, \bar{R}, Z, \delta, \omega, F_1, F_2)$. Clearly, $\Delta$ can be written in the following form

$$
\Delta = \bigcup_{a_k \in \Sigma} \delta_{a_k}
$$

where $\delta_{a_k} \subseteq Q \times Q$ for all $a_k \in \Sigma$. Hence, for all $a_k \in \Sigma$, using our formulas $(\ast)$ and $(\ast \ast)$, we obtain the upper transition functor $T_{a_k} : B \rightarrow M^Q$ and the lower transition functor $P_{a_k} : B \rightarrow M^Q$. It follows that we have functors $T_\delta = (T_{a_k})_{a_k \in \Sigma} : B \rightarrow (M^Q)^\Sigma$ and $P_\delta = (P_{a_k})_{a_k \in \Sigma} : B \rightarrow (M^Q)^\Sigma$. We state that $T_\delta$ is the labeled upper transition functor constructed by means of $\bar{F}$ and $P_\delta$ is the labeled lower transition functor constructed by means of $\bar{F}$. Note that any mapping $T : B \rightarrow (M^Q)^\Sigma$ corresponds to a mapping $\tilde{T} : \Sigma \times B \rightarrow M^Q$ so that, for all $a_k \in \Sigma$, $T = (\tilde{T}(a_k, \cdot))_{a_k \in \Sigma}$.

Hence, $T_\delta$ and $P_\delta$ will play the role of our transition functors. Now, let $P = (P_{a_k})_{a_k \in \Sigma} : B \rightarrow (M^Q)^\Sigma$ and $T = (T_{a_k})_{a_k \in \Sigma} : B \rightarrow (M^Q)^\Sigma$ be morphisms of partially ordered sets. For all $a_k \in \Sigma$, let $\Delta^{P_{a_k}}$ be the lower $P_{a_k}$-induced relation by $M$ and $\Delta_{T_{a_k}}$ be the upper $T_{a_k}$-induced relation by $M$. Then $\Delta^P = \bigcup_{a_k \in \Sigma} \Delta^{P_{a_k}}$ is called the lower $P$-induced state-transition relation and $\Delta_T = \bigcup_{a_k \in \Sigma} \Delta_{T_{a_k}}$ is called the upper $T$-induced state-transition relation. The general fuzzy automaton $\bar{F} = (Q, \Sigma, \bar{R}, Z, \delta, \omega, F_1, F_2)$ with state-transition relation $\Delta^P$ is said to be the lower $P_\delta$-induced general fuzzy automaton and we consider it as $\bar{F}^{P_\delta}$ and the general fuzzy automaton $\bar{F}$ with state-transition relation $\Delta_T$ is said to be the upper $T_\delta$-induced general fuzzy automaton and we consider it as $\bar{F}_{T_\delta}$. We say that the general fuzzy automaton $\bar{F}$ is recoverable from $T_\delta(P_\delta)$ if, for all $a_k \in \Sigma$, $\Delta$ is recoverable from $T_{a_k}(P_{a_k})$, i.e., if $\bar{F} = \bar{F}_{T_\delta}(\bar{F} = \bar{F}^{P_\delta})$.

**Theorem 3.1.** Let $M = ([0, 1]; \leq, 0, 1)$ be a non-trivial complete lattice,

$$
\bar{F} = (Q, \Sigma, \bar{R}, Z, \delta, \omega, F_1, F_2)
$$

be a general fuzzy automaton and $B$ be a bounded subposet of $M^Q$. Let $P_\delta : B \rightarrow (M^Q)^\Sigma$ and $T_\delta : B \rightarrow (M^Q)^\Sigma$ be labeled transition functors constructed by means of $\bar{F}$.

Then for all $b_1, b_2 \in B$,

(i) $P_B(b_1) \leq b_2 \iff b_1 \leq T_B(b_2)$.

Moreover, the following holds.

(ii) If $\Delta = \Delta_{T_\delta}$ and $T_\delta(B) \subseteq B^\Sigma$ then $\Delta = \Delta_{T_\delta} = \Delta^{P_\delta}$.

(iii) If $\Delta = \Delta^{P_\delta}$ and $P_\delta(B) \subseteq B^\Sigma$ then $\Delta = \Delta_{T_\delta} = \Delta^{P_\delta}$.

**Proof.** (i): Clearly for all $b_1, b_2 \in B$

$$
P_B(b_1) \leq b_2 \iff \forall a_k \in \Sigma, P_{a_k}(b_1) \leq b_2
$$

$$
\iff \forall a_k \in \Sigma, q_i \in Q, P_{a_k}(b_1)(q_i) \leq b_2(q_i)
$$

$$
\iff \forall a_k \in \Sigma, q_i \in Q, q_j \in Q_{\text{prec}}(q_i, a_k), b_2(q_i) \leq b_2(q_j)
$$

$$
\iff \forall a_k \in \Sigma, q_i \in Q, q_j \in Q_{\text{succ}}(q_i, a_k), b_1(q_i) \leq b_2(q_j)
$$

$$
\iff \forall a_k \in \Sigma, q_i \in Q, b_1(q_i) \leq T_{a_k}(b_2)(q_j)
$$

$$
\iff \forall a_k \in \Sigma, b_1 \leq T_{a_k}(b_2)
$$

$$
\iff b_1 \leq T_B(b_2).
$$
(ii): Assume that \( \Delta = \Delta_{T_b} \) and \( T_b(B) \subseteq B^2 \). We first show that \( \Delta^P \subseteq \Delta_{T_b} \). Let \( q_i, q_j \in Q \) and \( (q_i, q_j) \in \Delta^P \). Let \( b_1 \in B \). We put \( b_2 = T_b(b_1) \). Partially (i) we have \( P_b(T_b(b_1)) \leq b_1 \) and hence \( T_b(b_1)(q_j) \leq P_b(T_b(b_1))(q_j) \leq b_1(q_j) \), i.e., \( (q_i, q_j) \in \Delta_{T_b} \) and we have \( \Delta^P \subseteq \Delta_{T_b} \). Then \( \Delta \subseteq \Delta^P \subseteq \Delta_{T_b} = \Delta \) which yields the statement.

\[ \square \]

(iii): It follows from the same reasoning as in (ii).

**Example 3.2.** Consider the general fuzzy automaton \( \tilde{F} \) of Example 3.1. Let \( P \) be a restriction of the operator \( P_{\delta_b} \) of Example 3.1 and Let \( T \) be a restriction of the operator \( T_{\delta_b} \) of the same example. Let us compute \( \Delta_T \) and \( \Delta^P \). We have \( \Delta_T = \Delta^P = \{ (q_0, q_1), (q_2, q_2) \} \). Hence the transition relation \( \delta_b \) of Example 3.1 coincides with our induced transition relations \( \Delta_T \) and \( \Delta^P \). We can see from above that the operator \( T_{\delta_b} \) bears the maximal amount of information about the transition relation \( \delta_b \) on the suposet of \( P_{\delta_b} \circ T_{\delta_b} \). The same conclusion holds for the operator \( P_{\delta_b} \).

**Example 3.3.** Consider the general fuzzy automaton \( \tilde{F} \) of Example 3.1. Let us put \( B = [0, 1]^Q \). Let \( P : [0, 1]^Q \to [0, 1]^Q \) and \( T : [0, 1]^Q \to [0, 1]^Q \) be morphisms of partially ordered sets given as follows:

\[
\begin{align*}
T(0) &= 0, & P(0) &= 0, \\
T(0) &= s_0, & P(s_0) &= 0, \\
T(s_2) &= s_1, & P(s_1) &= s_2, \\
T(s_0) &= s_1, & P(s_2) &= s_0, \\
T(s_0) &= s_0, & P(s_0) &= s_0, \\
T(s_1) &= s_1, & P(s_1) &= s_0, \\
T(s_2) &= s_2, & P(s_2) &= s_2, \\
T(1) &= 1, & P(1) &= s_0.
\end{align*}
\]

Note that \( P \) coincides with the operator \( P_{\delta_b} \) of Example 3.1, and \( T \) coincides with the operator \( T_{\delta_b} \) of the same example. We have \( \Delta_T = \Delta^P = \{ (q_1, q_2), (q_2, q_1), (q_2, q_2) \} \). The transition relation \( \delta_a \) of Example 3.1 coincides with or induces transition relation \( \Delta_T \) and \( \Delta^P \).

The following corollary illustrates the situation in the case where our partially ordered set \( B \) of propositions is large enough, i.e., the case when \([0, 1]^Q \subseteq B\).

**Corollary 3.2.** Let \( M \) be a non-trivial complete lattice and \( \tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2) \) a general fuzzy automaton. Let \( B \) be a bounded suposet of \( M^Q \) so that \([0, 1]^Q \subseteq B\). Then the general fuzzy automaton \( \tilde{F} \) is recoverable both from \( P_{\delta_b} \) and \( T_{\delta_b} \).

**Proof.** Define, for all \( q_i \in Q \), an element \( b(q_i) \in [0, 1]^Q \subseteq B \) by

\[
[b(q_i)](q_j) = \begin{cases} 
0 & \text{if } q_i = q_j, \\
\mu^a(q_i) & \text{if } q_i \neq q_j.
\end{cases}
\]

Then \( b(q_i) \in B \) satisfies the assumption of part (ii) of Theorem 3.1, i.e., \( \Delta = \Delta_{T_b} \). Similarly, \( \Delta = \Delta_{P_b} \).

\[ \square \]

In the following theorems, we are going to demonstrate that the state-transition relation on \( Q \) and the transition operators on \( B \) form a Galois connection. This is significant since in every Galois connection one of its components completely ascerts the second one and vice versa.

Let \( M = ([0, 1]; \leq, 0, 1) \) be a non-trivial complete lattice and \( Q \) a non-empty set of states of \( \tilde{F} \). Let \( B \) be a bounded suposet of \( M^Q \), \( (Q \times Q); \leq, \emptyset, Q \times Q \) be the poset of all relations on \( Q \) and \( (\text{Map}(B, M^Q); \subseteq) \) be the poset of all order-preserving mappings \( T : B \to (M^Q)^2 \) so that \( T(1) = 1 \) and \( T_1 \subseteq T_2 \) if and only if \( T_2(b) \leq T_1(b) \) for all \( b \in B \). The smallest element of \( (\text{Map}(B, M^Q); \subseteq) \) is the constant mapping \( 1 \) so that \( 1(b) = (1, q_i) \) for all \( b \in B \). Let us put, for all \( \Delta_T \in (Q \times Q) \) and all \( T \in \text{Map}(B, M^Q) \), \( \phi(\Delta_T) = T_{\delta_b} \) and \( \psi(T) = \Delta_{T_b} \).

**Theorem 3.3.** Let \( M = ([0, 1]; \leq, 0, 1) \) be a non-trivial complete lattice and \( Q \) the set of states of general fuzzy automaton \( \tilde{F} \) so that \( B \) is a bounded suposet of \( M^Q \). Then the couple \((\phi, \psi)\) is a Galois connection between \( (Q \times Q); \leq, \emptyset, Q \times Q \) and \( (\text{Map}(B, M^Q); \subseteq) \).
Proof. It is clear that φ and ψ defined above are order-preserving mappings. It is enough to check that, for all $\Delta_T \in \xi(Q \times Q)$ and all $T \in \text{Map}(B, \mathcal{M}^Q)$

$$\phi(\Delta_T) \subseteq T$$ if and only if $\Delta_T \subseteq \psi(T)$.

Assume first that $\phi(\Delta_T) \subseteq T$ holds and let $(q_i, q_j) \in \Delta_T$. Then, for all $b \in B$: we have $T(b)(q_i) \leq (\phi(\Delta_T)(b))(q_i) = T_\delta(b)(q_i) = \land\{b(q_i)q_j \in Q_{\text{succ}}(q_i, a_k)\} \leq b(q_j)$. This yields that $(q_i, q_j) \in \psi(T) = \Delta_T$.

On the other hand, assume that $\Delta_T \subseteq \psi(T)$ and let $b \in B$, $q_i \in Q$. Either the set $\{q_j \in Q|(q_i, q_j) \in \Delta_T\}$ is empty in which case $(T_\delta(b))(q_i) = 1$ which yields $(T(b))(q_i) \leq 1 = (\phi(\Delta_T)(b))(q_i)$ or $(q_i, q_j) \in \Delta_T$ is not empty. In the last case, we have $\{b(q_i)q_j \in T|(q_i, q_j) \in \Delta_T\}$ is not empty and by the definition of $\phi(\Delta_T) = T_\delta$ we have $(T_\delta(b))(q_i) = \land\{b(q_i)q_j \in Q_{\text{succ}}(q_i, a_k)\} \leq b(q_j)$ for all $q_j \in Q$ so that $(q_i, q_j) \in \Delta_T$. Since $\Delta_T \subseteq \psi(T)$ we have, for all $q_j \in Q$ so that $(q_i, q_j) \in \Delta_T$, that, for all $c \in B$, $(T(c))(q_i) \leq c(q_j)$. It follows that $(T(c))(q_i) \leq (T_\delta(c))(q_j) = (\phi(\Delta_T)(c))(q_j)$.

But we have just proved that $\phi(\Delta_T) \subseteq T$.

\[\square\]

Remark 3.4. We indicate that our recoverable relations from the respective upper transition operators are exactly fixpoints of the composition $\psi \circ \phi : \xi(Q \times Q) \to \xi(Q \times Q)$.

Dually, let $M = ([0,1]; \leq, 0, 1)$ be a non-trivial complete lattice and $Q$ be the set of states of general fuzzy automata $\tilde{F}$. Let $A$ be a bounded subposet of $\mathcal{M}^Q$, $\xi(Q \times Q); \subseteq, 0, Q \times Q)$ be the poset of all relations on $Q$ and $(\text{Map}(A, \mathcal{M}^Q); \leq)$ be the poset of all order-preserving mappings $P : A \to (\mathcal{M}^Q)^Q \subseteq$ so that $P(0) = 0$ and $P_1 \leq P_2$ if and only if $P_1(a) \leq P_2(a)$ for all $a \in A$. The smallest element of $(\text{Map}(A, \mathcal{M}^Q); \leq)$ is the constant 0 so that $0(a) = 0(q) \subseteq Q \times Q)$ for all $a \in A$. Let us put, for all $\Delta_T \in \xi(Q \times Q)$ and all $P \in \text{Map}(A, \mathcal{M}^Q)$, $\Phi(\Delta_P) = P_\delta$ and $\Psi(P) = \Delta^P_{\tilde{F}}$.

Theorem 3.5. Let $M = ([0,1]; \leq, 0, 1)$ be a non-trivial lattice and $Q$ be the set of states of general fuzzy automata $\tilde{F}$ so that $A$ is a bounded subposet of $\mathcal{M}^Q$. Then the couple $(\Phi, \Psi)$ is a Galois connection between $(\xi(Q \times Q); \subseteq, 0, Q \times Q)$ and $(\text{Map}(A, \mathcal{M}^Q); \leq)$.

Proof. Consider $(\text{Map}(A, \mathcal{M}^Q); \leq) = (\text{Map}(A^{op}, (\mathcal{M}^Q)^Q); \subseteq), \Phi(\Delta_P) = P_\delta = \varphi(\Delta_{T^{-1}})$ and $\Psi(P) = \Delta^P_{\tilde{F}} = \psi(T^{-1})$. Then the proof is similar to that of Theorem 3.5.

Example 3.4. Consider the general fuzzy automaton $\tilde{F}$, the set of propositions $B$ and the state - transition relation $\Delta$ of Example 3.1. From Example 3.1, we know the labeled upper transition functor $T_\delta = (T_{\delta_b}, T_{\delta_h})$ and the labeled lower transition functor $P_\delta = (P_{\delta_b}, P_{\delta_h})$ from $B$ to $([0,1]^{Q^2})$. Since $B = [0,1]^Q$ we have $T_{\delta_b}(B) \cup T_{\delta_h}(B) \subseteq B$ and $P_{\delta_b}(B) \cup P_{\delta_h}(B) \subseteq B$. Now, we use $T$ for computing the transition relations, $\Delta_{T_{\delta_b}}$ and $\Delta_{T_{\delta_h}}$ (by the formula ($*$) and Example 3.4) and $P_h$ for computing the transition relations $\Delta^P_{\delta_b}$ and $\Delta^P_{\delta_h}$ (by the formula (**) and Example 3.4). We obtain by the Corollary 3.2 that $\Delta_{T_{\delta_b}} = \Delta^P_{\delta_b} = \delta_b$ and $\Delta_{T_{\delta_h}} = \Delta^P_{\delta_h} = \delta_h$. It follows that $\Delta_{T_\delta} = \Delta^P_{\delta_b} \cup \Delta_{T_{\delta_h}} = \Delta$, i.e., Our given state-transition relation $\Delta$ simultaneously is recoverable by the transition functors $T_\delta$ and $P_h$. Hence these functors are characteristic of the triple $(B, \Sigma, Q)$.

By a synthesis in theory of systems, it is usually meant that the task constructs a general fuzzy automaton $\tilde{F}$ which realizes a dynamic process at least partially known to the user. Hence, we are given a description of this dynamic process and we know the set $\Sigma$ of inputs. Our task is to set up the set $Q$ of states and a relation $\Delta$ on $Q$ labeled by elements from $\Sigma$ so that the constructed general fuzzy automaton $\tilde{F}$ induces the logic, i.e., the partially ordered set of propositions which corresponds to the presented descriptions. The algebraic tools which were collected in previous sections enable us to solve the mentioned task. In what follows, we represent a construction of $Q$ and $\Delta$ which provides our logic with the transition functor representing the dynamics of our system. As mentioned in the previous section, our logic $B$ will be considered to be a bounded subposet $B$ of a power $\mathcal{M}^Q$ where $M$ is a complete lattice of truth-values. Our logic $B$ is equipped with a transition functor $T : B \to (\mathcal{M}^Q)^Q$ where $\Sigma$ is a set of possible inputs. We ask that either $T = T_\delta$ or $T = P_h$. Depending on the respective type of our submitted logic and of the properties of $T$ we will introduce some possible solutions to this task.

For any bounded partially ordered set $B = (B; \leq, 0, 1)$, we have a full set $S_B$ of morphisms of bounded partially ordered set into the algebra regarded as a bounded partially ordered set $([0,1]; \leq, 0, 1)$. The elements $h_D : B \to [0,1]$ of $S_B$ (indexed by proper down-sets $D$ of $B$) are morphisms of bounded partially ordered sets defined by the prescription, for all $a_k \in \Sigma$

$$h_{D_{a_k}}(b) = \begin{cases} T_{\delta_b}(b) & \text{if } b \notin D \\ 0 & \text{if } b \in D \end{cases}$$
In other words, every bounded partially ordered set $B$ can be embedded into an algebra $[0,1]^S$ for a certain set $S$ via the mapping $\tau_{\Sigma}$.

Thus, it seems confident to apply the bounded partially ordered set $M = ([0,1]; \leq, 0, 1)$ for the construction of our state-transition $\Delta \subseteq S_B \times \Sigma \times S_B$. As it was mentioned in the beginning of this section, we are interested in a construction of a general fuzzy automaton $\hat{F}$ for a given set $\Sigma$ of inputs and determined by a certain partially ordered set of propositions. We cannot assume that this set of propositions is necessarily a Boolean algebra. In the previous part, we supposed that this logic $B$ is a bounded partially ordered set $B = (B; \leq, 0, 1)$. Now, we are going to solve the situation when it is only a subset $C$ of $B$.

**Theorem 3.6.** Let $B = (B; \leq, 0, 1)$ be a bounded partially ordered set so that $B$ is a bounded subposet of $M^{S_B}$.

Let $(C; \leq, 1)$ be a subposet of $B$ including $1$, and $\Sigma$ a non-empty set. Let $T = (T_{a_k})_{a_k \in \Sigma}$ where $T_{a_k} : C \to M^{S_B}$ are morphisms of partially ordered sets so that $T_{a_k}(1) = 1$ for all $a_k \in \Sigma$. Let $\Delta_T$ be the labeled upper $T$-induced state-transition relation and $T_b : B \to (M^{S_B})^\Sigma$ be the labeled upper transition functor constructed by means of the upper $T_b$-induced automaton $\hat{F}_{T_b}$. Then, for all $b \in C$, $T(b) = T_b(b)$.

**Proof.** Clearly, $T_b = (T_{b_k})_{a_k \in \Sigma}$ where $T_{b_k} : B \to M^{S_B}$ are morphisms of partially ordered sets for all $a_k \in \Sigma$.

We write $\Delta_T = \bigcup_{a_k \in \Sigma} \Delta_{T_{a_k}}$ where $\Delta_{T_{a_k}}$, $a_k \in \Sigma$ are the $T_{a_k}$-induced relation by $M$. Let us choose $b \in C$ and $a_k \in \Sigma$ arbitrarily, but fixed. We have to check that $T_{a_k}(b) = T_{b_k}(b)$. Suppose that $h_D \in S_B, a_k \in \Sigma$. It is enough to verify that $T_{a_k}(b)(h_D) = \lambda(h_D) | h_C \in Q_{suc}(h_D, a_k)$. Evidently, for all $h_C \in S_B$ so that $(h_D, h_C) \in \Delta_T$, $T_{b_k}(b)(h_D) \leq h(h_C)$. Hence $T_{a_k}(b)(h_D) \leq \lambda \{h(h_C) | h_C \in Q_{suc}(h_D, a_k)\}$. To get the other inequality, assume that

$$T_{a_k}(b)(h_D) < \lambda \{h(h_C) | h_C \in Q_{suc}(h_D, a_k)\}.$$ 

Then $T_{a_k}(b)(h_D) = 0$ and $\lambda \{h(h_C) | h_C \in Q_{suc}(h_D, a_k)\} \neq 0$. Put $V_{a_k} = \{z \in B \exists y \in C, T_{a_k}(y)(h_D) \neq 0 \text{ and } y \leq z\}$. It follows that $b \notin V_{a_k}$ and $V_{a_k}$ is an upper set of $B$ so that $1 \in V_{a_k}$ (since $h_{a_k}(1) = 0 \neq 0$). Let $W_{a_k}$ be a maximal proper upper set of $B$ including $V_{a_k}$ so that $b \notin W_{a_k}$. Put $U_{a_k} = B \setminus W_{a_k}$. Then $U_{a_k}$ is a proper down-set, $0 \in U_{a_k}$, $h_{a_k}(b) = 0$ and $h_{a_k}(z) \neq 0$ for all $z \in V_{a_k}$, i.e., $h_{a_k} \in S_B$ so that $T_{a_k}(a)(h_D) \leq a(h_{a_k})$ for all $a \in C$. But this yields $(h_D, h_{a_k}) \in \Delta_{T_{a_k}}$, i.e., $0 \neq \lambda \{h(h_C) | h_C \in Q_{suc}(h_D, a_k)\} \leq h_D(h_{a_k}) = 0$, a contradiction.

□

Using the relation $\Delta^P$ instead of $\Delta_T$, we can obtain a statement dual of Theorem 3.6.

Consequently, with respect to the above mentioned materials and Theorem 3.6, we obtain the the upper $T_b$-induced general fuzzy automaton $\hat{F}_{T_b}$ as ten-tuple machine denoted with $\hat{F}_{T_b} = (S_B, \Sigma, \bar{R} = \{(h_{b_0}, \mu_B(h_{b_0}))\}, Z, \omega, \delta, \bar{\delta}, T_b, F_1, F_2)$ where,

(i) $S_B$ is the set of states, $S_B = \{h_D : B \to [0,1], D \subseteq B\}$ so that for all $a_k \in \Sigma$

$$h_{D_{a_k}}(b) = \begin{cases} T_{b_k}(b) & \text{if } b \notin D \\ 0 & \text{if } b \in D, \end{cases}$$

(ii) $\Sigma$ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,

(iii) $\bar{R} = \{(h_{b_0}, \mu_B(h_{b_0}))\}$ is the set of fuzzy start state,

(iv) $Z$ is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,

(v) $\omega : S_B \to Z$ is the output function,

(vi) $\delta : S_B \times \Sigma \times S_B \to [0,1]$ is the transition function defined by:

$$\delta(h_D, a_k, h_C) = h_{D_{a_k}}(b) \lor h_{C_{a_k}}(b)$$

for all $b \in B$ and $D, C \subseteq B$,

(vii) $\delta : (S_B \times [0,1]) \times \Sigma \times S_B \to [0,1]$ is the augmented transition function so that:

$$\mu^{t+1}(h_D) = \delta((h_D, \mu^t(h_D)), a_k, h_C) = F_1(\mu^t(h_D), \delta(h_D, a_k, h_C)),$$

(viii) $T_b : B \to (M^{S_B})^\Sigma$ is the labeled transition function so that $T_{b_k}(b)(h_D) = \lambda \{b(h_C) | h_C \in Q_{suc}(h_D, a_k)\}$ for all $a_k \in \Sigma$,

(ix) $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function,

(x) $F_2 : [0,1]^* \to [0,1]$ is called multi-membership resolution function.

**Example 3.5.** Consider again the set $Q = \{q_0, q_1, q_2\}$ of states, the set $\Sigma = \{a, b\}$, and the set of propositions $B = [0,1]^Q$ of Example 3.1. Assume that $C = \{a_2, s'_0, s_1, 1\} \subseteq B$ from the logic $B$ of Example 3.1. Assume further that our partially known transition operator $T$ from $C$ to $([0,1]^Q)^\Sigma$ is given as follows:

$$T_0(0) = 0, \quad T_0(s'_0) = s'_0, \quad T_0(1) = 1,$$

$$T_1(0) = 1, \quad T_1(s'_0) = s'_0, \quad T_1(1) = 1.$$
Note that we have chosen $T$ as a restriction of the operator $T_{\delta}$ from Example 3.1 on the set $C$. Then, by an easy computation we obtain from (*) that $\Delta_T = \Delta_{T_a} \cup \Delta_{T_b}$ where $\Delta_{T_a} = \{(q_1, q_2), (q_2, q_1), (q_2, q_2)\}$ and $\Delta_{T_b} = \{(q_0, q_1), (q_2, q_2)\}$. From Theorem 3.6, we have that $T$ is a restriction of the operator $T_{\Delta_T}$ on the set $C$. Moreover, we can see that our state transition relation $\Delta$ from Example 3.1 coincides with the induced state-transition relation $\Delta_T$, i.e., our partially known transition operator $T$ has given us a full information about the general fuzzy automaton $\tilde{F}$ from Example 3.1.

4. Conclusion

By a general fuzzy automaton, we contributed to show a set of propositions related to a given automaton and that the truth-values are depended on the states, inputs and membership values of active states at time $t$. This approach enables us to consider automata from a different point of view which is more close to logical treatment and helps us make estimations about the behavior of automaton particularly in a nondeterministic mode. The logic consists of propositions on the given GFA and its dynamic nature is stated by means of the so-called transition functor. This logic enables us to derive a certain relation on the set of states labeled by inputs. In fact, we showed that if our set of propositions is large enough, this recovering of the transition relation is possible. Moreover, a very challenging implication of our approach is that a zero-weight transition is possible and is different from no transition. A zero-weight transition may give rise to the activation of a successor due to the activation of its predecessor. While in all types of conventional automata, a zero-weight transition means no transition, in our approach to general fuzzy automata a zero-weight transition does not necessarily imply no transition. That is why we use $[0; 1]$ as the fuzzy interval. Then, in this paper we studied the theory of general fuzzy automata by using the concepts of operators. Such operators help us in the algebraic study of general fuzzy automata theory and provide a platform to use fuzzy topological therein in the future.

Compliance with Financial and Ethical Standards

Funding: This study was not funded by any organization or institute.

Conflict of Interest: The authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

References

[27] E. S. Santos, Maximin automata, Information and Control, 12 (1968), 367-377.


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