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Original Article

On the gradient Finsler Yamabe solitons

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ABSTRACT: Here, it is proved that the potential functions of Finsler Yamabe solitons have at most quadratic growth in distance function. Also it is obtained a finite topological type property on complete gradient Finsler Yamabe solitons under suitable scalar curvature assumptions.

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1. Introduction

The Yamabe flow was introduced first by R.S. Hamilton to study Yamabe's conjecture, stating that any metric is conformally equivalent to a metric with constant scalar curvature, see [9]. Yamabe flow is an evolution equation on a Riemannian manifold (M, g) defined by

$$\frac{\partial g}{\partial t} = -Rg, \qquad g(t=0) := g_0,$$

where R is the scalar curvature. Under the Yamabe flow, the conformal class of metrics remains invariant and is expected to evolve a manifold toward one with constant scalar curvature. Let (M, g) be a Riemannian manifold, a quad (M, g, V, λ) is said to be a *Yamabe soliton* if g satisfies the equation

$$\mathcal{L}_V g = 2(\lambda - R)g, \tag{1.1}$$

where V is a smooth vector field on M, $\mathcal{L}_V$ the Lie derivative along V and λ a real constant. A Yamabe soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If the vector field V is a gradient of a potential function f, then (M, g, V, λ) is said to be *gradient* and (1.1) takes the familiar form

$$\nabla\nabla f = (\lambda - R)g.$$

Yamabe solitons are special solutions of the Yamabe flow and naturally arise as limits of dilations of singularities in the Yamabe flow. Solutions of Yamabe elliptic equation on Riemannian manifolds are laying in the Sobolev space $H_1^2(M)$. Currently, a natural extension of Sobolev spaces is defined on Finsler manifolds, see [2].

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J.Y. Wu has proved that the complete shrinking gradient Yamabe solitons with bounded scalar curvature have finite topological type, see [11]. Also, it is shown that a complete gradient shrinking Ricci soliton has finite topological type if its scalar curvature is bounded by Fang, Man and Zhang, see [8]. Next, it is proved that a complete non-compact shrinking Yamabe soliton has finite fundamental group and its first cohomology group vanishes under a suitable condition, see [5]. Recently, a natural extension of Yamabe solitons for Finsler metrics is considered and a similar result is obtained for the complete Finslerian Yamabe solitons, see [6], [5], [4], [3] and [7].

2. Preliminaries and terminologies

Let M be a real n-dimensional differentiable manifold. We denote by TM its tangent bundle and by $\pi : TM_0 \longrightarrow M$, the fibre bundle of non zero tangent vectors. A *Finsler structure* on M is a function $F : TM \longrightarrow [0, \infty)$, with the following properties: I. Regularity: F is C^∞ on the entire slit tangent bundle $TM_0 = TM\backslash 0$. II. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$. III. Strong convexity: The $n \times n$ Hessian matrix $g_{ij}(x, y) = ([\frac{1}{2}F^2]_{y^i y^j})$ is positive definite at every point of TM_0. A *Finsler manifold* (M, F) is a pair consisting of a differentiable manifold M and a Finsler structure F. The Hessian matrix g_{ij} define a $(0, 2)$-tensor field g on π^*TM, called Finslerian metric tensor. The formal Christoffel symbols of second kind and the spray coefficients are denoted respectively by $\gamma^i_{jk} := g^{is}\frac{1}{2}\big(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j}\big)$, and $G^i := \frac{1}{2}\gamma^i_{jk}y^j y^k$. We consider also the *reduced curvature tensor* R^i_k which is expressed entirely in terms of the x and y derivatives of spray coefficients G^i, see [1].

$$R^i_k := \frac{1}{F^2}\big(2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}\big). \tag{2.1}$$

In the general Finslerian setting, one of the Ricci tensors introduced by H. Akbar-Zadeh is defined by

$$Ric_{jk} := [\frac{1}{2}F^2\mathcal{R}ic]_{y^j y^k},$$

where $\mathcal{R}ic = R^i_i$ is called the *Ricci scalar* and R^i_k is defined by (2.1), (see [1] page 191). One of the advantages of this Ricci quantity is its independence to the Cartan, Berwald or Chern (Rund) connections. A family of Finsler metrics $g(t)$ on M is called a Finsler Yamabe flow if it satisfies the equations

$$\frac{\partial}{\partial t}g_{jk} = -H_g g_{jk}, \quad g(t=0) := g_0,$$

where $H_g = g^{ij}Ric_{ij}$ is called the scalar curvature. This equation implies that

$$\frac{\partial}{\partial t}(\log F(t)) = -\frac{1}{2}H_g, \quad F(t=0) := F_0,$$

where, F_0 is the initial Finsler structure corresponding to g_0. Let $\gamma : [a, b] \longrightarrow M$ be a piecewise C^∞ curve on (M, F) with the velocity $\frac{d\gamma}{dt} = \frac{d\gamma^i}{dt}\frac{\partial}{\partial x^i} \in T_{\gamma(t)}M$. The integral length $L(\gamma)$ is given by

$$L(\gamma) = \int_a^b F(\gamma, \frac{d\gamma}{dt})dt.$$

For $p, q \in M$, denote by $\Gamma(p, q)$ the collection of all piecewise C^∞ curves $\gamma : [a, b] \longrightarrow M$ with $\gamma(a) = p$ and $\gamma(b) = q$. Let M be a connected manifold and define a distance function $d : M \times M \longrightarrow [0, \infty)$ by

$$d(p, q) := \inf_{\gamma\in\Gamma(p,q)} L(\gamma).$$

Note that in general this distance function is not symmetric, see [1]. According to the Hopf-Rinow's theorem, on a forward (or backward) geodesically complete Finsler space, every two points $p, q \in M$ can be joined by a minimal geodesic.

The map $\gamma(s, t)$ admits a canonical lift defined by

$$\hat{\gamma}(s, t) := (\gamma(s, t), \gamma'(s, t)).$$

Denote by SM the sphere bundle, defined by $SM := \bigcup_{x\in M} S_xM$ where $S_xM := \{y \in T_xM | F(x, y) = 1\}$. For a vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ on M define $\|X\|_x = \max_{y\in S_xM} \sqrt{g_{ij}(x, y)X^i X^j}$, where $x \in M$, (see [1] at p. 321). Since S_xM is compact, $\|X\|_x$ is well defined.

3. Gradient Finsler Yamabe solitons

Let $\rho : M \to \mathbb{R}$ be a real differentiable function on the Finsler manifold (M, g). We consider here the vector field $grad\rho(p) \in T_pM$, defined by $grad\rho := \rho^i(x)\frac{\partial}{\partial x^i}$, where $\rho^i(x) = g_{ij}(x, grad\rho(x))\frac{\partial \rho}{\partial x^j}$ as the gradient of ρ at point $p \in M$. Equivalently

$$g_{grad\rho(p)}(X, grad\rho(p)) = d\rho_p(X), \qquad \forall X \in T_pM.$$

Let (M, F) be a Finsler manifold and $V = v^i(x)\frac{\partial}{\partial x^i}$ a vector field on M. We call the quad (M, F, V, λ) a Finsler *Yamabe soliton* if g_{jk} the Hessian related to the Finsler structure F satisfies

$$\mathcal{L}_{\hat{V}} g_{jk} = 2(\lambda - H)g_{jk}, \tag{3.1}$$

where, $\hat{V}$ is the complete lift of V and $\lambda \in \mathbb{R}$. A Finsler Yamabe soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If the vector field V is gradient of a potential function f, then (M, F, V, λ) is said to be gradient Finsler Yamabe soliton. The Yamabe soliton is said to be forward complete (resp. compact) if (M, F) is forward complete (resp. compact). Note that according to the Hopf-Rinow's theorem, two notions forward complete and forward geodesically complete are equivalent.

3.1. Estimations for the potential function

We want to obtain some estimation for the potential function as follows:

Theorem 3.1. *The potential function $f : M \longrightarrow \mathbb{R}$ of a complete non-compact gradient Finsler Yamabe soliton has at most quadratic growth for distance function. That is,*

(a) if $H \geqslant 0$, $f(x) \leqslant \frac{\lambda}{2}d(p, x)^2 + Cd(p, x) + D$,

(b) if $H \geqslant \lambda$, $f(x) \leqslant Cd(p, x) + D$.

Proof. Recall that the Lie derivative of a Finsler metric tensor g_{jk} is given by

$$\mathcal{L}_{\hat{V}} g_{jk} = \nabla_j v_k + \nabla_k v_j + 2(\nabla_0 v^l)C_{ljk}, \tag{3.2}$$

where $\hat{V}$ is the complete lift of a vector field $V = v^i(x)\frac{\partial}{\partial x_i}$ on M, ∇ is the Cartan h-covariant derivative, C_{ljk} are the components of Cartan torsion tensor, $\nabla_0 := y^p\nabla_p$ and $\nabla_p := \nabla_{\frac{\delta}{\delta x^p}}$. Now by using $V = \nabla f$ in the gradient Finsler Yamabe soliton (3.1), we have

$$\nabla_j\nabla_k f + \nabla_k\nabla_j f + 2(\nabla_0\nabla^l f)C_{ljk} = 2(\lambda - H)g_{jk}. \tag{3.3}$$

Fix one point $p \in M$. Given any point $x \in M$, let γ be a unit speed minimal geodesic joining p to x and $\rho(x) = d(p, x)$. Contracting (3.3) with ${\gamma'}^j{\gamma'}^k$ gives along γ

$$\nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'} f = \lambda - H. \tag{3.4}$$

On the other hand, by compatibility of metric in the Cartan connection, we have along the geodesic γ

$$\nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'} f = \nabla_{\hat{\gamma}'}({\gamma'}^k\nabla_k f) = \frac{d}{ds}({\gamma'}^k\nabla_k f) = \frac{d}{ds}(<\gamma', \nabla f>), \tag{3.5}$$

where $\hat{\gamma}' = {\gamma'}^j\frac{\delta}{\delta x^j}$. Therefore we have for all $t \in [0, \rho(x)]$

$$\int_0^t \nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'} f \, ds = <\gamma'(t), \nabla f> - <\gamma'(0), \nabla f> . \tag{3.6}$$

(a) By assumptions $H \geqslant 0$, we have

$$\nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'} f = \lambda - H \leqslant \lambda.$$

Integrating both sides of the last equation leads to

$$\int_0^t \nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'} f \, ds \leqslant \lambda t.$$

Replacing (3.6) in the last formula we get $<\gamma'(t), \nabla f> - <\gamma'(0), \nabla f> \leqslant \lambda t$ and therefore

$$\frac{d}{dt} f(\gamma(t)) \leqslant \lambda t + <\gamma'(0), \nabla f> .$$

By means of Cauchy-Schwarz inequality we have

$$\frac{d}{dt}f(\gamma(t)) \leqslant \lambda t + \|\nabla f\|_p.$$

Integrating from 0 to $\rho(x)$ leads to

$$f(\gamma(\rho(x))) - f(\gamma(0)) \leqslant \frac{\lambda}{2}\rho(x)^2 + (\|\nabla f\|_p)\rho(x).$$

Hence we get

$$f(x) \leqslant \frac{\lambda}{2}\rho(x)^2 + (\|\nabla f\|_p)\rho(x) + f(p),$$

as we have claimed in (a).
(b) $H \geqslant \lambda$, we have

$$\nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'}f = \lambda - H \leqslant 0.$$

Integrating both sides of the last equation leads to

$$\int_0^t \nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'}f \; ds \leqslant 0.$$

Replacing (3.6) in the last formula we get $<\gamma'(t), \nabla f> - <\gamma'(0), \nabla f> \leqslant 0$ and therefore

$$\frac{d}{dt}f(\gamma(t)) \leqslant <\gamma'(0), \nabla f> .$$

By means of Cauchy-Schwarz inequality we have

$$\frac{d}{dt}f(\gamma(t)) \leqslant \|\nabla f\|_p.$$

Integrating from 0 to $\rho(x)$ leads to

$$f(\gamma(\rho(x))) - f(\gamma(0)) \leqslant (\|\nabla f\|_p)\rho(x).$$

Hence we get

$$f(x) \leqslant (\|\nabla f\|_p)\rho(x) + f(p),$$

as we have claimed in (b). □

3.2. Finite topological type of gradient Finsler Yamabe solitons

Recall that M has finite topological type if M is homeomorphic to the interior of a compact manifold with boundary. Isotopy Lemma says if there is a proper smooth function $f : M \longrightarrow \mathbb{R}$ such that f has no critical points outside a compact subset of M, by Morse theory, M is diffeomorphic to the interior of a compact manifold with (smooth) boundary, see [10].

Theorem 3.2. *Let (M, F) be a geodesically complete Finsler manifold satisfying*

$$\mathcal{L}_{\hat{V}}g_{jk} \geqslant 2(\lambda - H)g_{jk}, \tag{3.7}$$

where $V = \nabla f$ and the scalar curvature $H \leqslant \Lambda < \lambda$. Then M has finite topological type.

Proof. It's enough to show that the potential function f is proper and has no critical points of a compact set. Let $p \in M$ be a fix point and let γ be a minimal geodesic by arc length joining p to any point $x \in M$. Note that $\rho(x) = d(p, x)$. Then we have

$$\nabla_{\hat{\gamma}'}\nabla_{\hat{\gamma}'}f \geqslant \lambda - H.$$

Integrating of both sides the last formula and using (3.6), we get

$$<\gamma'(t), \nabla f> - <\gamma'(0), \nabla f> \geqslant \int_0^t (\lambda - H)ds \geqslant \int_0^t (\lambda - \Lambda) = (\lambda - \Lambda)t.$$

Therefore

$$<\gamma'(t), \nabla f> \geqslant (\lambda - \Lambda)t + <\gamma'(0), \nabla f> .$$

By Cauchy-Schwarz inequality we have

$$\|\nabla f\|_{\gamma(t)} \geqslant (\lambda - \Lambda)t - \|\nabla f\|_p.$$

Set $t = \rho(x)$. we get

$$\|\nabla f\|_x \geqslant (\lambda - \Lambda)\rho(x) - \|\nabla f\|_p.$$

Therefore $\|\nabla f\|_x$ has a linear growth in $\rho(x) = d(p, x)$. Obviously, $f^{-1}\big((-\infty, a]\big)$ is compact for any $a < \infty$ and so f is a proper function. Also, one can easily check that f has no critical points outside of a compact set. In fact, it's enough to consider a compact set $\bar{\mathcal{B}}_p^+(\frac{2\|\nabla f\|_p}{\lambda-\Lambda})$. The deformation lemma(Isotopy Lemma) of Morse theory leads to M has finite topological type. □

Corollary 3.3. *Any forward complete shrinking, steady or expanding gradient Finsler Yamabe soliton with $H \leqslant \Lambda$ for some constant $\Lambda < \lambda$ has finite topological type.*

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