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Lie group analysis for short pulse equation

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ABSTRACT: In this paper, the classical Lie symmetry analysis and the generalized form of Lie symmetry method are performed for a general short pulse equation. The point, contact and local symmetries for this equation are given. In this paper, we generalize the results of H. Liu and J. Li [2], and add some further facts, such as an optimal system of Lie symmetry subalgebras and two local symmetries.

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1. Introduction

Non-linear PDEs arising in many applied fields like Biology, Fluid Mechanics, Plasma Physics and Optics, systems of impulse and Neural Networks, etc. and exhibit a rich variety of non-linear phenomena. The investigation of the exact solutions plays an important role in the study of non-linear systems. In this paper, we find Lie point symmetries, third order local symmetries, an optimal system of these two type symmetries, and corresponding invariant solutions for a general short pulse equation:

$$\text{SPE} \;:\; u_{xt} = \alpha u + \frac{\beta}{3}(u^3)_{xx}, \tag{1}$$

where $u = u(x,t)$ is the unknown real function and subscripts denote differentiation w.r.t. x and t; α and β are non-zero real parameters.

This general SPE was derived by T. Schafer and C.E. Wayne [8, p.94] as a model equation describing the propagation of ultra-short light pulses in silica optical fibres. In [8, 7], many results are obtained about the special SPE:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \tag{2}$$

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2. Lie contact and point symmetries

Let $J^1 = J^1(\mathbb{R}^2, \mathbb{R})$ be the jet space with coordinates (x, t, u, u_x, u_t). Let

$$\mathbf{v} = \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u + \eta^x\,\partial_{u_x} + \eta^t\,\partial_{u_t}, \tag{3}$$

be an infinitesimal Lie contact symmetry of (1), where ξ, τ and η are smooth functions $J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}$, and

$$Q = \xi u_x + \tau u_t - \eta, \tag{4}$$

be the characteristic function of (4). Thus

$$\begin{aligned} &\xi = Q_{u_x}, && \tau = Q_{u_t}, && \eta = u_x Q_{u_x} + u_t Q_{u_t} - Q, \\ &\eta^x = -Q_x - u_x Q_u, && \eta^t = -Q_t - u_t Q_u. && \end{aligned} \tag{5}$$

Then, the (3) is an infinitesimal Lie contact symmetry of the (1) if and only if

$$\mathbf{v}^{(2)}\Big(u_{xt} - \alpha u - \frac{1}{3}\beta(u^3)_{xx}\Big) = 0, \qquad u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx}, \tag{6}$$

where $\mathbf{v}^{(2)}$ is the second prolongation of $\mathbf{v}$:

$$\mathbf{v}^{(2)} = Q\,\partial_u + \mathrm{D}_x Q\,\partial_{u_x} + \mathrm{D}_t Q\,\partial_{u_t} + \mathrm{D}_x^2 Q\,\partial_{u_{xx}} + \mathrm{D}_x \mathrm{D}_t Q\,\partial_{u_{xt}} + \mathrm{D}_t^2 Q\,\partial_{u_{tt}}, \tag{7}$$

where D_x and D_t are total derivative w.r.t x and t, respectively.

By substituting $u_{x,t}$ from second equation of (7) in the first equation, we find a polynomial of u_{xx} and u_{tt} with functional coefficients of (x, t, u, u_x, u_t). Its coefficients must be zero:

$$\begin{aligned} &Q_{u_x,u_x} = 0, \quad Q_{u_x,u_t} = 0, \quad Q_{u_t,u_t} = 0, \\ &u_t Q_{u,u_x} + \alpha u Q_{u_x,u_x} + Q_{t,u_x} = 0, \quad \alpha u Q_{u_t,u_t} + u_x Q_{u,u_t} + Q_{x,u_t} = 0, \\ &u_x^2 Q_{u,u} + 2u_x Q_{x,u} + Q_{xx} + \alpha u\big(\alpha Q_{u_t,u_t} + 2u_x Q_{u,u_t} + 2Q_{x,u_t}\big) = 0, \\ &u_t Q_{u,u_t} - 5u_x Q_{u,u_x} - 5Q_{x,u_x} + Q_{t,u_t} - 4u Q_{u_x,u_t} - 2u_x Q_u = 0, \\ &u_x u_t Q_{u,u} + u_x Q_{t,u} + u_t Q_{x,u} + Q_{x,t} \\ &\quad + \alpha u\big(u_t Q_{u,u_t} + Q_{t,u_t} + Q_{x,u_x} + Q_u\big) + \alpha\big(u_x Q_{u_x} + u_t Q_{u_t} - Q\big) = 0. \end{aligned} \tag{8}$$

After solving the determining system (8), one finds that

$$Q = c_1\,u_x + c_2\,u_t + c_3\,(xu_x - tu_t - 3u); \tag{9}$$

where, c_1, c_2 and c_3 are arbitrary constants. Therefore,

Theorem 1. *The SPE (1) has a $3-$dimensional Lie algebra $\mathfrak{g}$ of point symmetries, generated by the infinitesimal generators*

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = x\,\partial_x - t\,\partial_t + 3u\,\partial_u, \tag{10}$$

and commutating table

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$-\mathbf{v}_2$
$\mathbf{v}_3$	$-\mathbf{v}_1$	$\mathbf{v}_2$	0

(11)

The SPE (1) has not any non-point contact symmetry.

3. Invariant solutions and its classification

The one-parameter groups G_i generated by the base of $\mathfrak{g}$ are as follows:

$$\begin{aligned} G_1 &: \exp(\varepsilon\mathbf{v}_1)\cdot(x, t, u) = (x + \varepsilon, t, u), \\ G_2 &: \exp(\varepsilon\mathbf{v}_2)\cdot(x, t, u) = (x, t + \varepsilon, u), \\ G_3 &: \exp(\varepsilon\mathbf{v}_3)\cdot(x, t, u) = \big(e^{\varepsilon}x, e^{-\varepsilon}t, e^{3\varepsilon}u\big), \end{aligned} \tag{12}$$

where ε is a real number.

Since each group G_i is a symmetry group of SPE (1) and if $u = f(x, y)$ is a solution of the SPE (1), so are the following functions

$$u = f(x+\varepsilon, t), \quad u = f(x, t+\varepsilon), \quad u = f\big(e^{\varepsilon}x, e^{-\varepsilon}t, e^{-3\varepsilon}u\big), \tag{13}$$

where ε is an arbitrary real number. Thus, for the arbitrary combination $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \in \mathfrak{g}$, the SPE (1) has the following solution:

$$u = f\big(e^{\varepsilon_3}x + \varepsilon_1, e^{-\varepsilon_3}t + \varepsilon_2, e^{-3\varepsilon_3}u\big), \tag{14}$$

where ε_i are arbitrary real numbers.

Let G be the symmetry Lie group of SPE (1). Now G operates on the set of solutions S of SPE (1), and $s \cdot G$ be the orbit of s, and H be a subgroup of G. Invariant $H-$solutions $s \in S$ are characterized by equality $s \cdot S = \{s\}$. If $h \in G$ is a transformation and $s \in S$, then

$$h \cdot (s \cdot H) = (h \cdot s) \cdot (hHh^{-1}). \tag{15}$$

Consequently, every invariant $H-$solution s transforms into an invariant $hHh^{-1}-$solution (Proposition 3.6 of [5]). Therefore, different invariant solutions are found from similar subgroups of G. Thus, the classification of invariant $H-$ solutions is reduced to the problem of the classification of subgroups of G, up to similarity. An optimal system of $s-$dimensional subgroups of G is a list of conjugacy inequivalent $s-$dimensional subgroups of G with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s-$dimensional sub-algebras forms an optimal system if every $s-$dimensional sub-algebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathbf{h}} = \mathrm{Ad}(g)\cdot\mathbf{h}$. Let H and $\tilde{H}$ be connected, $s-$dimensional Lie subgroups of the Lie group G with corresponding Lie sub-algebras $\mathbf{h}$ and $\tilde{\mathbf{h}}$ of the Lie algebra $\mathfrak{g}$. Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only $\tilde{\mathbf{h}} = \mathrm{Ad}(g)\cdot\mathbf{h}$ are conjugate sub-algebras (Proposition 3.7 of [5]). Thus, the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of sub-algebras, and so we concentrate on it.

For the one-dimensional sub-algebras, the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional sub-algebra is determined by a nonzero vector in Lie algebra symmetries of SPE (1) and so to "simplify" it as much as possible. The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots, \tag{16}$$

where $i, j = 1, \cdots, 3$. Let $F_i^{\varepsilon} : \mathfrak{g} \to \mathfrak{g}$ defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v})$, for $i = 1, \cdots, 3$. Therefore, if $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \in \mathfrak{g}$, then

$$\begin{aligned} F_i^{\varepsilon_1}(\mathbf{v}) &= (c_1 + \varepsilon_1 c_3)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3, \\ F_i^{\varepsilon_2}(\mathbf{v}) &= c_1\mathbf{v}_1 + (c_2 + \varepsilon_2 c_3)\mathbf{v}_2 + c_3\mathbf{v}_3, \\ F_i^{\varepsilon_3}(\mathbf{v}) &= e^{-\varepsilon_3}c_1\mathbf{v}_1 + e^{\varepsilon_3}c_2\mathbf{v}_2 + c_3\mathbf{v}_3. \end{aligned} \tag{17}$$

Applying these transformations, one can show that

Theorem 2. *A one-dimensional optimal system of $\mathfrak{g}$ is*

$$\mathbf{v}_1 + a\mathbf{v}_2, \quad b\mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{v}_3, \tag{18}$$

where a and b are real constants; and, a two-dimensional optimal system of $\mathfrak{g}$ is given by

$$\mathbf{v}_1, \mathbf{v}_2, \quad \mathbf{v}_1, \mathbf{v}_3, \quad \mathbf{v}_2, \mathbf{v}_3. \tag{19}$$

4. Local symmetries of SPE

One can generalize one-parameter Lie groups of point transformations with infinitesimal generators in the characteristic form $\mathbf{v} = Q(x, t, u, u_x, u_t)\,\partial_u$ to one-parameter s-order local transformations with infinitesimal generators of the form

$$\mathbf{v} = Q(x, t, u, \partial u, \partial^2 u, \cdots, \partial^s u)\,\partial_u, \tag{20}$$

where the infinitesimal components depend on derivatives of u up to some finite order $s \geq 1$. The prolongation of $\mathbf{v}$ is given by

$$\mathbf{v}^{(\infty)} = Q\,\partial_u + \mathrm{D}_x Q\,\partial_{u_x} + \mathrm{D}_t Q\,\partial_{u_t} + \mathrm{D}_x^2 Q\,\partial_{u_{xx}} + \mathrm{D}_x \mathrm{D}_t Q\,\partial_{u_{xt}} + \mathrm{D}_t^2 Q\,\partial_{u_{tt}} + \cdots. \tag{21}$$

where D_x and D_t are total derivative w.r.t x and t, respectively [1].

Then, for $s = 3$, (21) is an infinitesimal local symmetry of the (1) if and only if

$$\begin{aligned} &\mathbf{v}^{(\infty)}\Big(u_{xt} - \alpha u - \frac{1}{3}\beta(u^3)_{xx}\Big) = 0, && u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx}, \\ &u_{x^2t} = \mathrm{D}_x\big(\alpha u + \frac{1}{3}\beta(u^3)_{xx}\big), && u_{xt^2} = \mathrm{D}_t\big(\alpha u + \frac{1}{3}\beta(u^3)_{xx}\big), \\ &\qquad\cdots\cdots\cdots && u_{xttt} = \mathrm{D}_t^2\big(\alpha u + \frac{1}{3}\beta(u^3)_{xx}\big), \end{aligned} \tag{22}$$

which leads to a polynomial of u_{x^5} and u_{t^5}, with functional coefficients of

$$Q(x, t, u, u_x, u_t, u_{xx}, u_{tt}, , u_{xxx}, u_{ttt}, u_{x^4}, u_{t^4}) \tag{23}$$

and its derivatives. All of its coefficients must be zero. This leads to a system of 5 linear determining PDEs:

$$\begin{aligned} &\beta^4 u_{xx}^8 Q_{u_{t^4}^2} + Q_{u_{x^4}, u_{t^4}} = 0, \\ &\cdots\cdots\cdots \\ &u_x u_{tt} Q_{u,u_t} + \cdots + u_x u_{t^4} Q_{u,u_{ttt}} = 0. \end{aligned} \tag{24}$$

Therefore, the most general third-order characteristic function Q is

$$\begin{aligned} Q = (c_1\,t + c_2)\,u_t + 3c_1 u - c_1\,x u_x + c_3\,u_{ttt} - c_3\,\beta^3 u_{xx}^6 u_{xxx} \\ - \frac{3}{2} c_3\,\alpha\beta^2 u_x u_{xx}^4 - \big(c_3\,\beta\alpha^2 u_x^2 - c_5\big)\,u_x + \frac{c_4\,u_{xxx}}{\sqrt{2\beta u_{xxx}^2 + \alpha}}, \end{aligned} \tag{25}$$

where $c_1, \cdots, c_5$ are arbitrary constants. There is not any non-trivial second or fourth-order characteristics. Thus, we prove that

Theorem 3. *The most general third-order infinitesimal local symmetry generator of SPE* (1) *is a* $\mathbb{R}-$*linear combination of following five vector fields* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *of* (10) *and*

$$\begin{aligned} \mathbf{v}_4 &= \frac{u_{xxx}}{\sqrt{2\beta u_{xxx}^2 + \alpha}}\,\partial_u, \\ \mathbf{v}_5 &= \Big(u_{xxx} - \beta^3 u_{xx}^6 u_{xxx} - \frac{3}{2}\alpha\beta^2 u_x u_{xx}^4 - \alpha^2\beta u_{xxx}\Big)\,\partial_u. \end{aligned} \tag{26}$$

There is not any non-trivial second or fourth-order infinitesimal local symmetry generators.

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