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Original Article

Counting closed billiard paths

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ABSTRACT: Given a pool table enclosing a set of axis-aligned rectangles, with a total of n edges, this paper studies *closed billiard paths*. A closed billiard path is formed by following the ball shooting from a starting point into some direction, such that it doesn't touch any corner of a rectangle, doesn't visit any point on the table twice, and stops exactly at the starting position. The *signature* of a billiard path is the sequence of the labels of edges in the order that are touched by the path, while repeated edge reflections like *abab* are replaced by *ab*. We prove that the length of a signature is at most $4.5n - 9$, and we show that there exists an arrangement of rectangles where the length of the signature is $1.25n + 2$. We also prove that the number of distinct signatures for fixed shooting direction ($45°$) is at most $1.5n - 6$.

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1. Introduction

The *closed billiard paths* problem was first presented at the open problem session of the 29th Canadian Conference on Computational Geometry, and to our best knowledge, this paper gives the first study on this problem. O'Rourke defines the problem as follows [2]. Consider a collection of axis-aligned rectangles, which all are enclosed in one outer rectangle; the outer rectangle is the enclosing boundary (billiard table) and other rectangles are inner boundaries (obstacles). A *simple, closed billiard path* is a path that is closed, non-self-intersecting forming a simple polygon, that do not touch a rectangle corner, where all reflections are mirror reflection. The *signature* of a billiard path is defined by the labels of edges which reflect the billiard path, where the repeated edge reflections $(ab)^k$ are reduced to ab. In Figure 1, the signature of the billiard path $\langle 4(56)^2 1292(37)^2 348\rangle$ is 456129237348.

For a set of rectangles with a total of n edges, O'Rourke posed the following open questions on simple, closed billiard paths.

1. What is the maximum length of such a signature?
2. What is the largest number of distinct signatures achievable for one fixed reflection angle (*e.g.,* $45°$ in Figure 1).
3. What is the largest number of distinct signatures achievable for paths at arbitrary reflection angles?

In this paper, in Section 2, we prove an upper bound of $4.5n - 9$ for the length of a signature, and we show that there exists an arrangement of rectangles with signature length $1.25n + 2$. Also, in Section 3, we prove that the number of distinct signatures for fixed reflection angle $45°$ is bounded by $1.5n - 6$.

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2. Maximum Length of a Signature

Consider a billiard path P and its signature S. We define an alternative signature S' from S, where any pattern aba in S is replaced by ab; this results in $|S| \leq 1.5 \cdot |S'|$. In Figure 1, the alternative signature of the billiard path with signature $S = 456129237348$ is $S' = 4561293748$.

To compute the maximum length of a signature, we construct a planar graph $G(V, E)$ from a billiard path $P = \langle \dots p_{k-1} p_k p_{k+1} \dots \rangle$ such that $|E| = |S'|$, where S' is the alternative signature of P. The set V of nodes of G are defined as follows: Corresponding to each rectangle edge e_i we place a node v_i at the middle of e_i (see Figure 2(b)). We also draw the set E of edges (v_i, v_j) of G using the following rules.

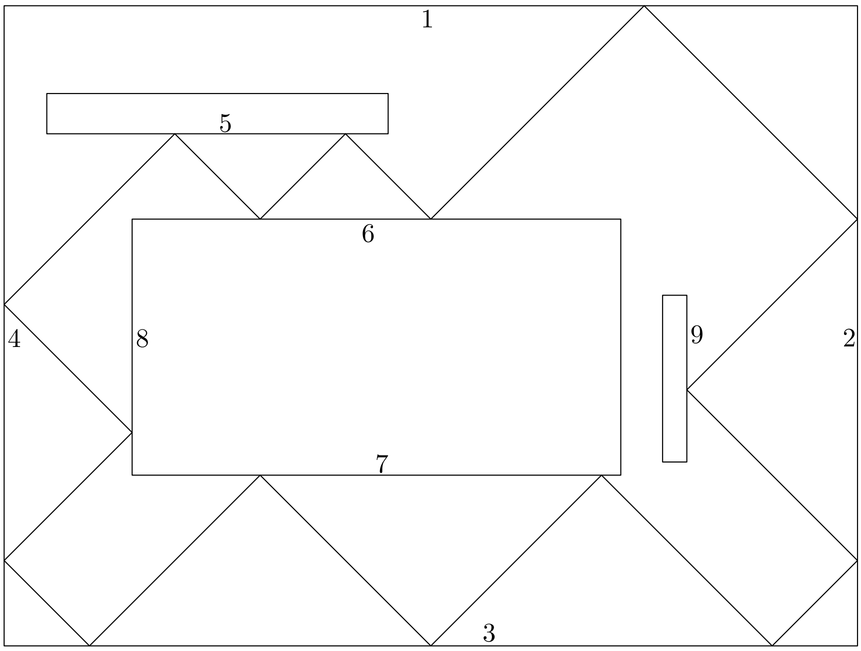

Figure 1: A billiard path of a set of four rectangles.

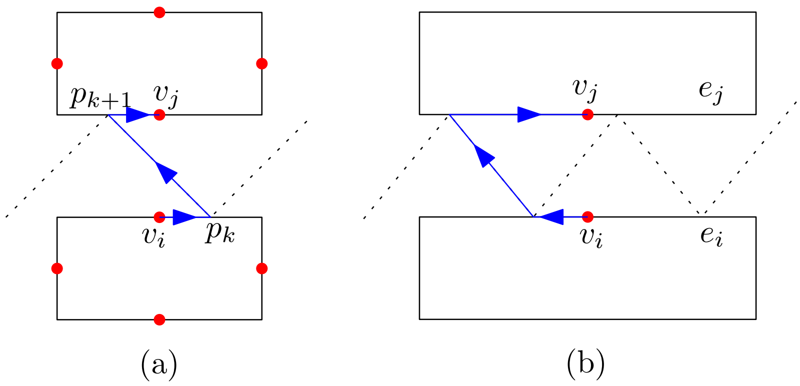

Figure 2: Consecutive segments between e_i and e_j; (a) only one segment. (b) more than one segment.

(a) If there is only one segment $\overline{p_k p_{k+1}}$ in P between rectangle edges e_i and e_j, where $p_k \in e_i$ and $p_{k+1} \in e_j$, then we draw the edge (v_i, v_j) with two bends at p_k and p_{k+1}, *i.e.*, $(v_i, v_j) = \langle \overline{v_i p_k}, \overline{p_k p_{k+1}}, \overline{p_{k+1} v_j} \rangle$. See Figure 2(a).

(b) If there exists a sequence of consecutive segments $\overline{p_k p_{k+1}}, \overline{p_{k+1} p_{k+2}}, \dots$ in P between two parallel rectangle edges e_i and e_j, we ignore all but the first one ($\overline{p_k p_{k+1}}$) and draw an edge (v_i, v_j) in the graph G, similar to case (a).

(c) If there are two *non-consecutive* segments $\overline{p_k p_{k+1}}$ and $\overline{q_k q_{k+1}}$ in P between two *horizontal* rectangle edges e_i and e_j, where $p_k, q_k \in e_i$ and $p_{k+1}, q_{k+1} \in e_j$ and $|p_k|_x < |q_k|_x$, then there exists at least one rectangle inside the quadrilateral $p_k, p_{k+1}, q_{k+1}, q_k$. Here, $|r|_x$ denotes the x-coordinate of r. Take the leftmost rectangle inside the quadrilateral and place a node v_l on its left edge e_l; see Figure 3.

Note that there is no segment in path P touching e_l, otherwise it would cause a self-intersection which means that P is not a valid path. Also note that there can be no two segments in P both above and below e_l for the same reason; following this observation, w.l.o.g., assume there is no segment of P between e_l and e_j.

Corresponding to $\overline{p_k p_{k+1}}$, we draw an edge $(v_i, v_j) = \langle \overline{v_i p_k}, \overline{p_k p_{k+1}}, \overline{p_{k+1} v_j} \rangle$ with two bends at p_k and p_{k+1}, and also corresponding to $\overline{q_k q_{k+1}}$ we draw an edge (v_l, v_j) with one bend at b, the intersection of e_j and the vertical line through v_l; *i.e.*, $(v_l, v_j) = \langle \overline{v_l b}, \overline{b v_j} \rangle$.

Our method can easily be extended if there exist more than two, say three, non-consecutive segments (denote by $p_k p_{k+1}$, $q_k q_{k+1}$, and $r_k r_{k+1}$ in order from left to right) in P between two horizontal rectangle edges e_i and e_j: all we need to do is adding a new edge (v'_l, v_j) in G corresponding to the next segment $r_k r_{k+1}$, where v'_l is on the left edge e'_l of the leftmost rectangle in the quadrilateral $q_k q_{k+1} r_{k+1} r_k$.

A similar approach works if the *non-consecutive* segments are between two *vertical* rectangle edges e_i and e_j. Note that if there are two segments in P between a vertical rectangle edge and a horizontal rectangle edge, as shown in Figure 4, this leads to self-intersection in P and is thus obsolete.

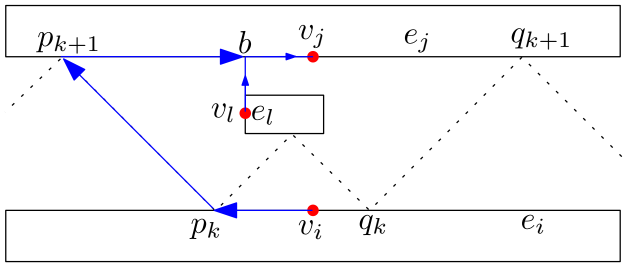

Figure 3: Two non-consecutive segments $\overline{p_k p_{k+1}}$ and $\overline{q_k q_{k+1}}$ between e_i and e_j.

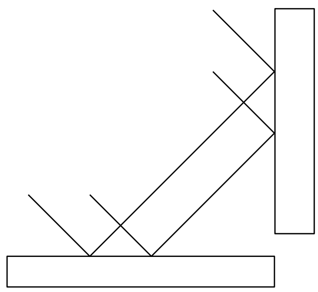

Figure 4: Two segments in path between a vertical and a horizontal rectangle edge lead to self-intersection.

By the above construction, there could be the case that some segments s_i and s_l (like $s_l = \overline{b v_j}$ and $s_i = \overline{p_{k+1} v_j}$ in Figure 3) of edges (v_i, v_j) and (v_l, v_j) overlap on an edge e_j of a rectangle. We can define a valid ordering between the edges of G whose segments overlap on some rectangle edge e_j: Consider a line L_j parallel to e_j which intersects all the edges of G overlapping on e_j; see Figure 5. The distances from the intersection points to v_j define an ordering for the corresponding edges (and hence an ordering for their segments which overlap on e_j; denote the segments in order by $s_{j,1}, s_{j,2}, \ldots, s_{j,k}$). Since we have an ordering for segments $s_{j,1}, s_{j,2}, \ldots, s_{j,k}$, it is easy to slightly move the segments in order (by adding some new bends) inside the rectangle of e_j in such a way that no two edges of G overlap. Thus we can obtain a planar drawing for G. Figure 5 depicts a drawing of three edges inside a rectangle with no overlappings of their segments.

By Euler's formula [1], it is proven that for any planar graph G, the number of edges is at most $3n - 6$. Thus $|S'| \leq 3n - 6$. Since the length of a signature S of a billiard path P is at most 1.5 times the length of the alternative signature S' of P, the following obtains.

Theorem 1. *For a collection of axis-aligned rectangles, all enclosed in one rectangle, with a total of n edges, the maximum length of a signature is at most $4.5n - 9$.*

Remark. In Theorem 1 there is no dependency on the reflection angle. Therefore, such a bound holds for arbitrary reflection angles.

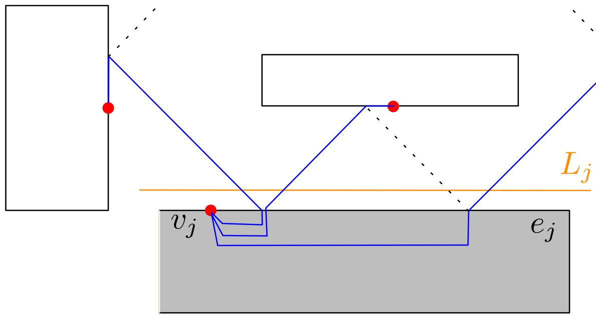

Figure 5: Drawing the edges with no overlappings.

Lower bound. Consider a set of $4k + 2$ rectangles with a total of n edges. Let R be the outer rectangle, where we place the bottom-left corner of R at the origin and its top-right corner at $(4k + 3, 8)$.

Let $R_1, \ldots, R_{k+1}$ be a collection of rectangles, each with width 1 and height 7; place the rectangle R_i with its center at $(4i - 2.5, 4)$. Let $R'_1, \ldots, R'_k$ be a collection of rectangles, each with width 2 and height 2; place the rectangle R'_i with its center at $(4i - 0.5, 4)$. Let $R''_1, \ldots, R''_{2k}$ be a collection of rectangles, each with width 1 and height 2; place the rectangle R''_{2i+1} with its center at $(4i - 0.5, 1.5)$ and the rectangle R''_{2i} with its center at $(4i - 0.5, 6.5)$.

If we shoot a ray from $(1, 0)$ with reflection angle $45°$ to the right, length of the signature of the resulting billiard path would be $1.25n + 2$. As seen in Figure 6, for $k = 2$, the number of edges is equal to $n = 40$ and the length of the signature is 52. It is obvious to see that, by increasing k by one unit (*i.e.*, adding four new inner rectangles) the signature length increases by 20.

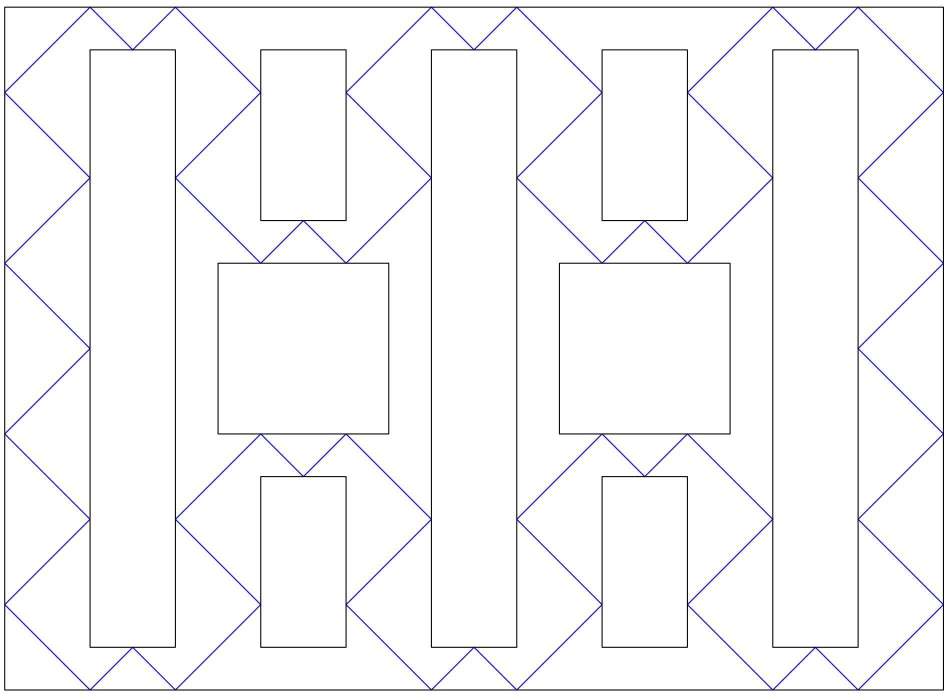

Figure 6: The signature length of ten rectangles is 50.

3. Number of Distinct Signatures

To count the number of distinct signatures, we charge each signature to a corner of a rectangle.

For each biliard path P, we define a starting point p_s residing on some upper edge of a rectangle [1]. Now, you can trace the path by following a ray of light shooting from p_s, where the edges of rectangles are perfect mirrors[2].

[1] If there is no upper edge involved in the path P, rotate the whole collection of rectangles by $90°$ until you find such upper edge; if you still don't find such an upper edge, let p_s be some reflection point on the bottom edge of the bounding rectangle.

[2] The ray shooting is the problem of determining the first intersection of a ray with a set of obstacles [1].

Since we have a fixed reflection angle 45°, the billiard path P can now be represented by the *shooting point* p_s residing on some upper edge of a rectangle.

Lemma 1. *Let P be a billiard path with signature S and shooting point p_s . If p_s is translated to the right by the amount α such that the corresponding rays do not touch any corner over translation, the new point p'_s (which is of distance α to p_s) is a shooting point for a billiard path P' with the same signature $S' = S$.*

Proof. Let $P = \langle p_1 \dots p_k \rangle$ and $P' = \langle p'_1 \dots p'_k \rangle$, where $p_1 = p_s$ and $p'_1 = p'_s$. We claim, for $1 \leqslant i \leqslant k$, that the points p_i and p'_i touch the same edge (resulting in the same signature) and are within distance α.

We know that p_s and p'_s are at distance α. Also, p_s and p'_s reside on the same edge, as otherwise there would be a point from p_s at distance $\beta < \alpha$ such that it would touch a corner (the contradiction).

Assume that for all p_1 to p_i and p'_1 to p'_i, our claim is true, and p_i and p'_i reside on some upper edge (if not, rotate the whole collection of rectangles until it satisfies). Let $p_j = p_{i+1}$ and $p'_j = p'_{i+1}$. It is easy to see that $\|p_j - p'_j\|_\infty = \|p_i - p'_i\|_\infty = \alpha$ (see Figure 7): If p_j and p'_j both reside on a horizontal edge, then we have $d(p_j, p'_j) = \alpha$ (because $p_j p'_j$ is parallel to $p_i p'_i$). If p_j and p'_j both reside on a vertical edge, then $d(p_j, p'_j) = \alpha \cdot \tan(45°) = \alpha$. Thus we have proven our claim. □

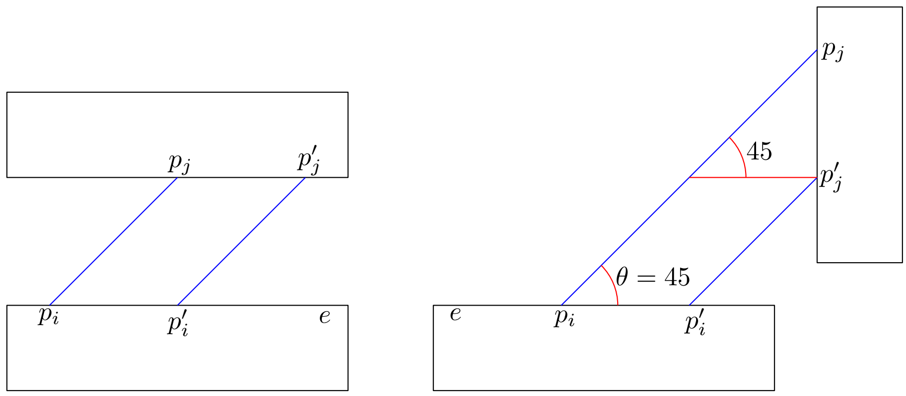

Figure 7: Shooting rays to the right.

Note that (as shown in Figure 8) there are three cases if a ray touches the corner of some rectangle.

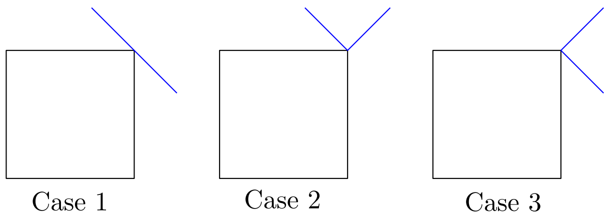

Figure 8: Three cases when a ray touching the corner of some rectangle.

Lemma 2. *For each signature, there exists a path with the same signature touching a corner of a rectangle.*

Proof. By translating the shooting point p_s of a path P slightly to *right*, unless we touch a corner of a rectangle, our signature remains the same (from Lemma 1).

Over translation, if a corner touches, which is one of the three cases in Figure 8, we still can have the same signature as before. If Case 2 (resp. Case 3 and Case 1) occurred when touching the corner, and we move the path more to the right (resp. up and left/down), then the corresponding signature changes. Therefore, we can charge any path P and its signature to (one of the three cases of) some corner. In Figure 9, the translated path can touch three corners p_1 (Case 2), p_2 (Case 1), and p_3 (Case 3) with the same signature as before.

Note that it not possible that a translated path touches a corner of the outer rectangle *before* a corner of an inner rectangle. If there exists such a path (see the blue dashed path in Figure 10), then a bit before touching the corner of the outer rectangle the path must lead to self-intersection, which is not a valid billiard path.

If the translated path touches a corner of the outer rectangle, where at the same time it touches a corner of an inner rectangle, then the path reflects to itself (a degenerate case); we interpret this case which will have the

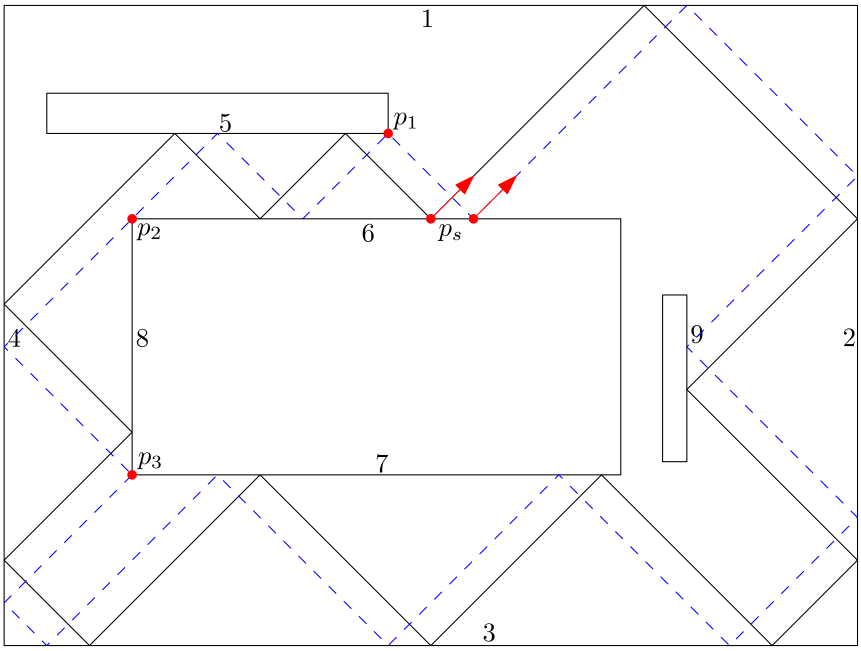

Figure 9: The dashed blue path is the translation of the billiard path.

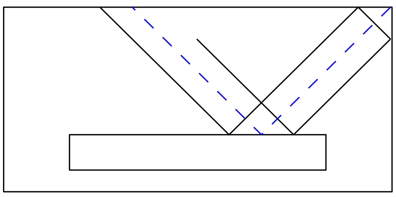

Figure 10: A path touches a corner of the outer rectangle before a corner of an inner rectangle.

same signature as before. Consider the billiard path in Figure 11 (with signature $S = \ldots 13124 \ldots$). We can say that the translated path first reflects to the rectangle side 3 (Case 2), next touches both rectangle sides 1 and 2, respectively, at the cornet of the outer rectangle, and finally reflects to the rectangle side 4 (Case 3). This implies that both Case 2 and Case 3 occur at the same time (and we say that the signature of the translated path is still $S = \ldots 13124 \ldots$). ☐

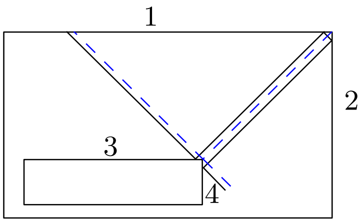

Figure 11: A translated path touches both a corner of the outer rectangle and a corner of an inner rectangle.

From above lemma, the following results:

Corollary 1. *There exists a rectangle corner from which a shooting ray can produce the maximum-length signature.*

Theorem 2. *For a collection of axis-aligned rectangles, all enclosed in one rectangle, with a total of n edges, there are at most $1.5n - 6$ distinct signatures.*

Proof. From Lemma 2, over translation of a path to the *right*, each signature S can be charged to a corner (one of the three cases, in Figure 8) of a rectangle. Thus we have at most $3(n - 4)$ distinct signatures.

Note that by translating the path in the opposite direction (*i.e.*, translating to the *left*), without changing the signature S, the path touches another corner. This implies that each signature is counted twice. Thus the number of distinct signatures is at most $3(n - 4)/2$. ☐

4. Discussion

For a set of axis-aligned rectangles with a total of n edges, we gave an upper bound $4.5n - 9$ for the maximum length of a signature, showed an arrangement of rectangles with signature length $1.25n + 2$, and proved that the number of distinct signatures is at most $1.5n - 6$. It would be interesting to improve these bounds. Also, another interesting question is to provide an efficient algorithm to find the maximum-length signature.

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