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The bimodal standard normal density and kurtosis

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ABSTRACT: In this article, first a density by the name "The bimodal standard normal density" is introduced and denoted by $\mathbf{b}\boldsymbol{\phi}(z)$. Then, a definition for the kurtosis of bimodal densities relative to $\mathbf{b}\boldsymbol{\phi}(z)$ is presented. Finally, to illustrate the introduced kurtosis, a few examples are provided and a real data set is studied, too.

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1. Introduction

It is well-known that the kurtosis for a unimodal density $f(x)$, of a random variable X, is $\frac{\mu_4}{\sigma^4}$, where $\mu = E(X), \sigma^2 = E(X-\mu)^2$, and $\mu_4 = E(X-\mu)^4$. This parameter was introduced by K. Pearson in a 1905 Biometrika paper, only for unimodal densities. It is supposed to measure the peakedness or flatness of a density relative to the standard normal density $\phi(z)$. The primary aim of this article is to suggest a kurtosis measure for a continuous bimodal density relative to a bimodal normal density called "the bimodal standard normal density" denoted by $\mathbf{b}\boldsymbol{\phi}(z)$. The article is organized in the following manner. In Section 2, we consider a bimodal symmetric normal density. The bimodal standard normal density $\mathbf{b}\boldsymbol{\phi}(z)$ is introduced in Section 3. In Section 4, the modes of $\mathbf{b}\boldsymbol{\phi}(z)$ are found. Section 5 is devoted to a definition of kurtosis. In addition, a few examples are considered.

2. A bimodal symmetric normal density

Consider the following normal densities, with $d > 0$,

$$g(x \pm d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x \pm d)^2}, \tag{2.1}$$

and the symmetric mixed normal density

$$m(x; d) = \frac{1}{2} g(x+d) + \frac{1}{2} g(x-d). \tag{2.2}$$

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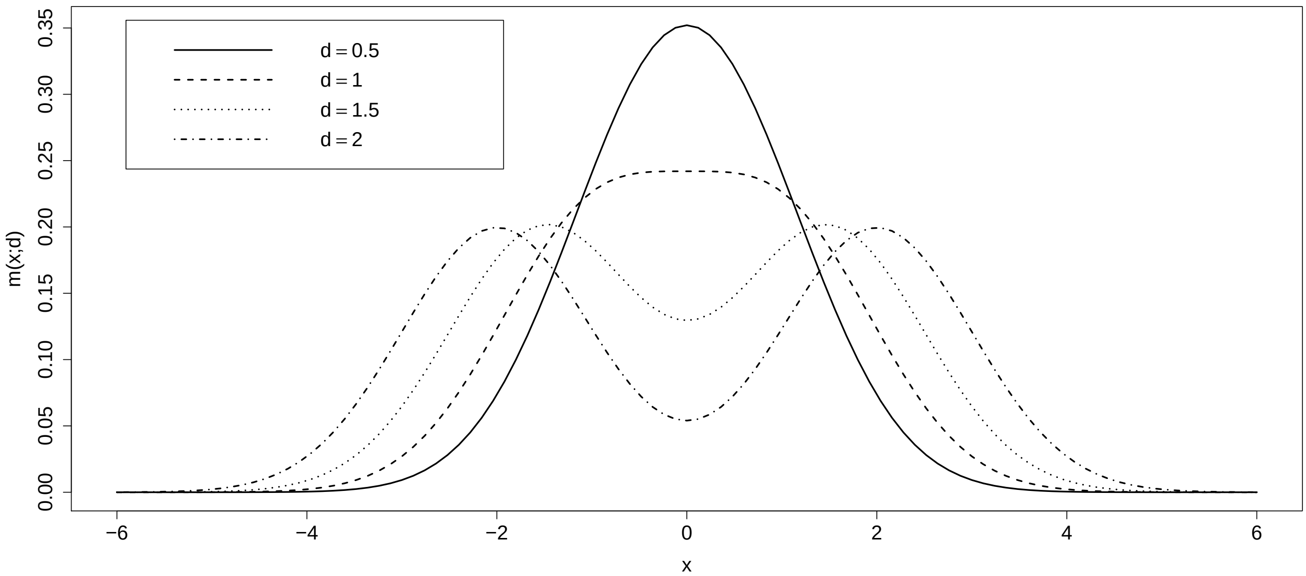

Figure 1: Graphs of $m(x; d)$ for $d = 0.5, 1, 1.5, 2$.

Theorem 2.1. *Density (2.2) is unimodal if $d \leq 1$ and bimodal if $d > 1$.*

Proof: By taking the first and second derivatives from (2.1), we have

$$g'(x \pm d) = -(x \pm d)g(x \pm d),, \tag{2.3}$$
$$g''(x \pm d) = [(x \pm d)^2 - 1]g(x \pm d). \tag{2.4}$$

Now, using (2.2), (2.3) and (2.9), we obtain

$$m'(x; d) = \frac{1}{2}[-(x + d)g(x + d)] + \frac{1}{2}[-(x - d)g(x - d)],$$
$$m''(x; d) = \frac{1}{2}[(x + d)^2 - 1]g(x + d) + \frac{1}{2}[(x - d)^2 - 1]g(x - d).$$

The only root of $m'(x; d) = 0$ is $x = 0$ and $m''(0, d) = g(d)(d^2 - 1) < 0$ when $d < 1$. If d=1, then $m''(0, 1) = 0$, but $m^{(4)}(0, 1) < 0$. Hence, $m(x; d)$ is unimodal with mode (abscissa of maximum point) zero, when $d \leq 1$.
Now, suppose that $d > 1$. Since $m''(0, d) > 0$, zero, minimizes the symmetric continuous function $m(x; d)$, satisfying $\lim_{x \to \pm\infty} m(x; d) = 0$. Therefore, $m(x; d)$ is bimodal with two modes and one demode (abscissa of minimum point) zero (see Figure 1).

3. The standard bimodal normal density

In this section, we try to standardized $m(x; d)$, $d > 1$, given in Section 2. Let $X \sim m(x; d), d > 1$. It is clear that $E(X) = 0$ and $var(X) = E(X^2) = 1 + d^2 = \sigma^2$. The density of the standard random variable $Y = (X - 0)/\sigma$ is

$$f(y; d) = \frac{\sigma}{2}g(\sigma y + d) + \frac{\sigma}{2}g(\sigma y - d),$$

where $g(y \pm d)$ is given by (2.10). We denote the modes of $f(y; d)$ by $\pm M$, $(M > 0)$ and demode by $m = 0$.
For the standard normal density $\phi(0) = 1/\sqrt{2\pi} = 0.3990$. Now, we find d such that we also have $f(M; d) = 1/\sqrt{2\pi}$.
In the next section, by using Newton method, which is written by the Maple program, we find $d = 1.7260$, $\sigma = \sqrt{(1 + d^2)} = 1.9947$ and $M = 0.8607$. Thus, we obtain the bimodal standard normal density and we denote it by

$$\mathbf{b}\boldsymbol{\phi}(z) = \frac{\sigma}{2\sqrt{2\pi}}e^{-(\sigma z + d)^2/2} + \frac{\sigma}{2\sqrt{2\pi}}e^{-(\sigma z - d)^2/2}.$$

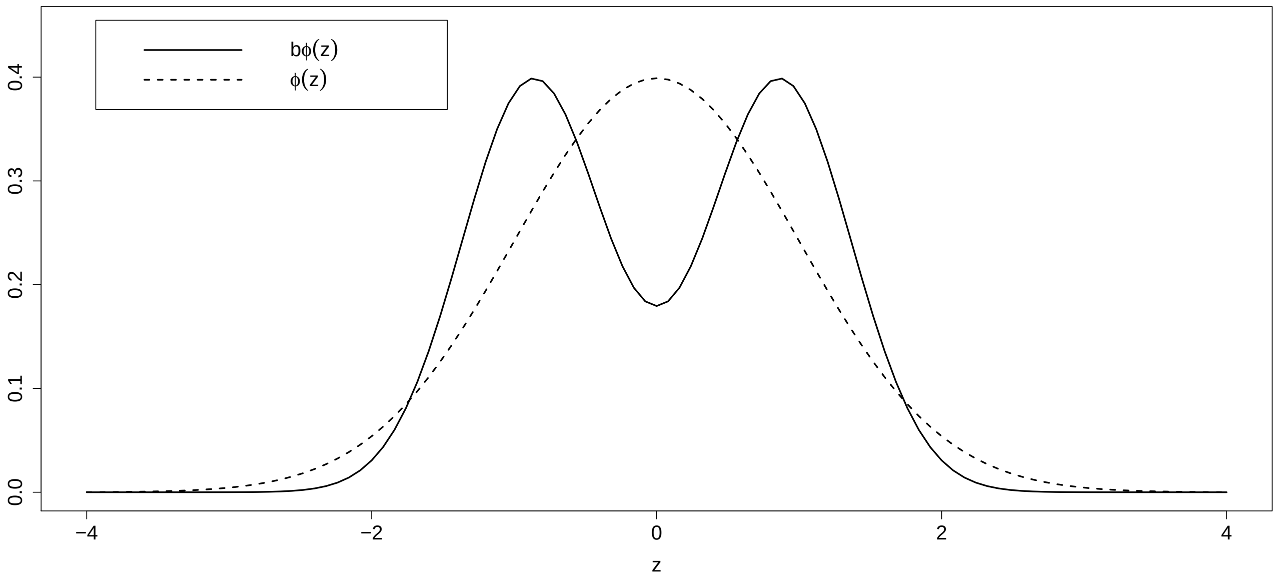

Figure 2: The graphs of $\phi(z)$ and $\mathbf{b}\boldsymbol{\phi}(z)$.

Figure 2 shows the graph of $\mathbf{b}\boldsymbol{\phi}(z)$ versus the graph of $\phi(z)$.

4. Computation of the modes of $\mathbf{b}\boldsymbol{\phi}(z)$

In this section, we find d and M for $\mathbf{b}\boldsymbol{\phi}(z)$, by a system of two equations with two unknowns. In summary, if $Z \sim \mathbf{b}\boldsymbol{\phi}(z)$, Table 1 contains the numerical features of $\mathbf{b}\boldsymbol{\phi}(z)$. The nonzero roots of derivative of $\mathbf{b}\boldsymbol{\phi}(z)$ are the two modes $\pm M$. Since $\mathbf{b}\boldsymbol{\phi}(\pm M) = 1/\sqrt{2\pi}$, by some simple algebra, we obtain the following equation

$$e^{-\frac{1}{2}(\sigma M+d)^2} + e^{-\frac{1}{2}(\sigma M-d)^2} = 2/\sigma, \tag{4.1}$$

where $\sigma = \sqrt{1+d^2}$ and $d > 1$. On the other hand, from the derivative of $\mathbf{b}\boldsymbol{\phi}(z)$ at $\pm M$, we have

$$\frac{d+\sigma M}{d-\sigma M} = e^{2d\sigma M}. \tag{4.2}$$

Solving (4.1) and (4.2) by Maple Program, we obtain $d = 1.7260$, $M = 0.8607$ (after 20 iterations), $\mathbf{b}\boldsymbol{\phi}(\pm M) = 1/\sqrt{2\pi} = 0.3990$ and $\mathbf{b}\boldsymbol{\phi}(0) = 0.1794$.

Table 1: Numerical features of $\mathbf{b}\boldsymbol{\phi}(z)$.

d=1.7260	mode factor
$\sigma = 1.9947$	scale
$M = \pm 0.8607$	modes
$m = 0$	demode
$\mathbf{b}\boldsymbol{\phi}(\pm M) = 1/\sqrt{2\pi} = 0.3990$	ordinate of modes
$\mathbf{b}\boldsymbol{\phi}(0) = 0.1794$	ordinate of demode
$E(Z) = 0$	expectation
$var(Z) = 1$	variance
$\boldsymbol{\kappa} = 0.2892$	kurtosis for $\mathbf{b}\boldsymbol{\phi}(z)$

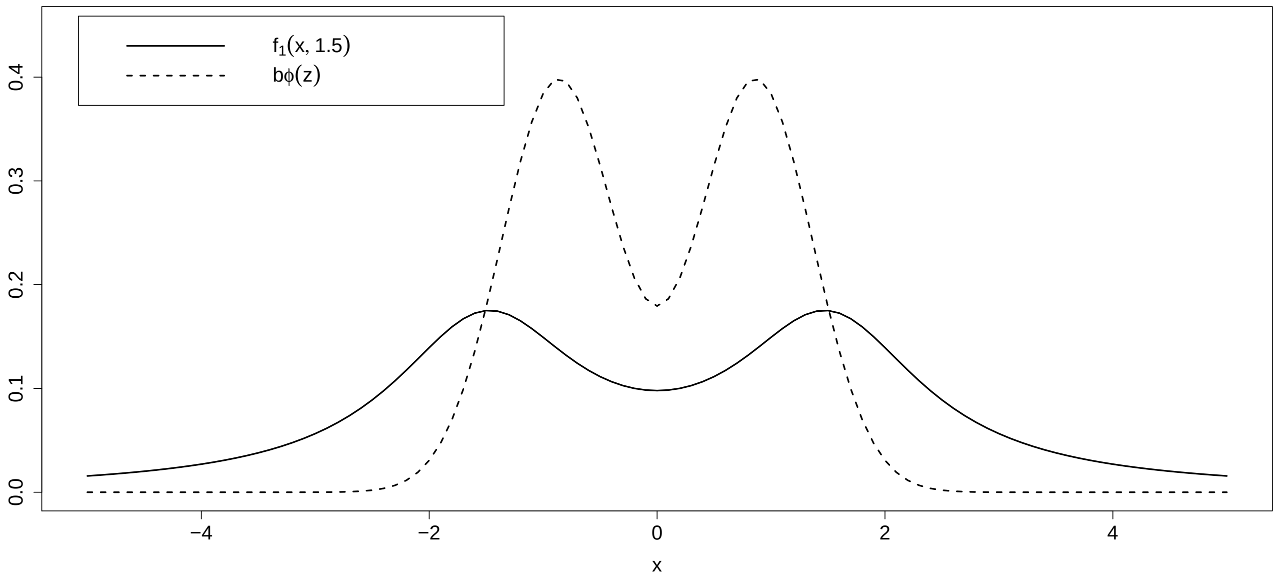

Figure 3: Graph of $f_1(x; 1.5)$ and $\mathbf{b}\boldsymbol{\phi}(z)$.

5. A definition for kurtosis of bimodal densities

Let $f(x)$ be a bimodal continuous density, for which $f'(x)$ and $f''(x)$ exist, with modes $M_1 < M_2$ and demode (abscissa of the minimum point of $f(x)$) m.

Definition of kurtosis:
The left kurtosis of $f(x)$ is defined by $L = [f(M_1) + f(m)]/2$ and the right by $R = [f(M_2) + f(m)]/2$.
This is a plausible definition, even for unimodal densities. Because of the fact that for $M_1 = M_2 = m = M$, $f(x)$ becomes close to a unimodal density and $L = R = [f(M) + f(m)]/2 = f(m)$.
For $\mathbf{b}\boldsymbol{\phi}(z)$, which is symmetric, we obtain (see Table 1)

$$L = R = \frac{\mathbf{b}\boldsymbol{\phi}(M) + \mathbf{b}\boldsymbol{\phi}(0)}{2} = \frac{0.3990 + 0.1794}{2} = 0.2892.$$

Therefore, the kurtosis for $\mathbf{b}\boldsymbol{\phi}(z)$ is the constant $\boldsymbol{\kappa} = 0.2892$.
If for a bimodal density $f(x)$, $L < \boldsymbol{\kappa} = 0.2892 (> \boldsymbol{\kappa} = 0.2892)$, the left side is flat (peaked) relative to $\mathbf{b}\boldsymbol{\phi}(z)$. Similarly,if $R < \boldsymbol{\kappa} = 0.2892 (> \boldsymbol{\kappa} = 0.2892)$, the right side is flat (peaked) relative to $\mathbf{b}\boldsymbol{\phi}(z)$. Now, we look at a few examples.

Example 1: We consider a symmetric mixture of two Cauchy densities

$$f_1(x; \alpha) = \frac{1}{2}\frac{1}{\pi(1 + (x + \alpha)^2)} + \frac{1}{2}\frac{1}{\pi(1 + (x - \alpha)^2)}.$$

This density is bimodal if $\alpha = 1.5$ and the features of the density are given in Table 2.

Table 2: Numerical features of a symmetric mixture of two Cauchy densities.

M_1	M_2	m
-1.4691	1.4691	0
$f_1(M_1; 1.5)$	$f_1(M_2; 1.5)$	$f_1(m; 1.5)$
0.1752	0.1752	0.0979

Since $L = R = 0.2365 < \boldsymbol{\kappa} = 0.2892$, therefore, $f_1(x; 1.5)$ is flat relative to $\mathbf{b}\boldsymbol{\phi}(z)$.(See Figure 3)

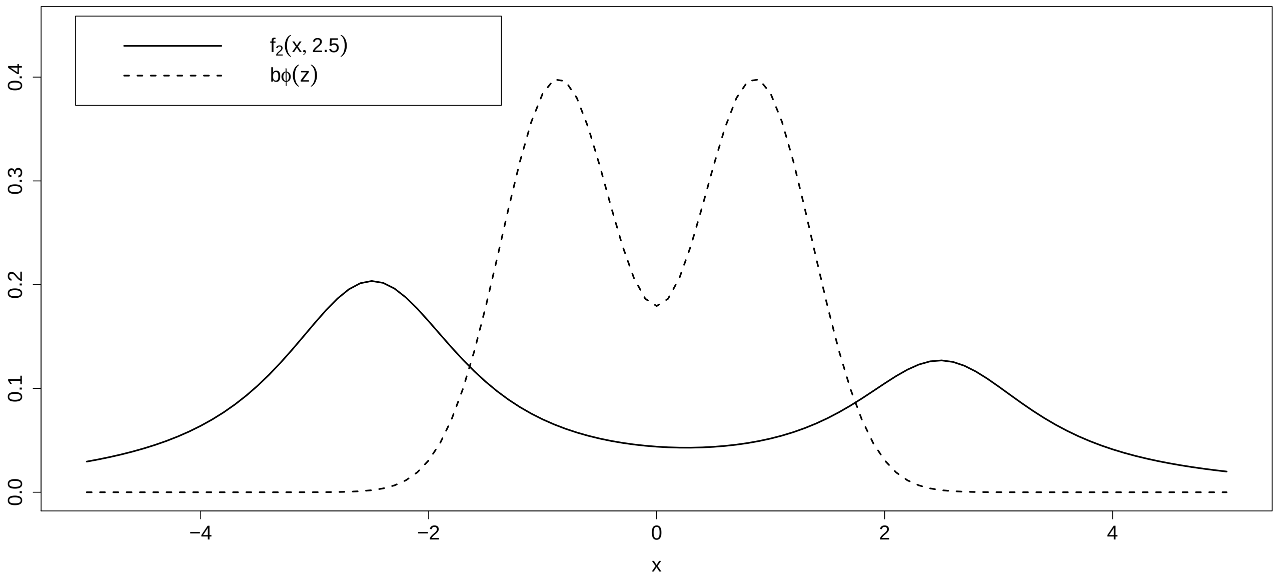

Figure 4: Graph of $f_2(x; 2.5)$ and $\mathbf{b}\boldsymbol{\phi}(z)$.

Example 2: In this example, we consider a non-symmetric mixture of two Cauchy densities

$$f_2(x;\alpha) = \frac{5}{8}\frac{1}{\pi(1+(x+\alpha)^2)} + \frac{3}{8}\frac{1}{\pi(1+(x-\alpha)^2)}.$$

$f_2(x; 2.5)$ is bimodal with some features given in Table 3.
For this density $L = 0.1232$ and $R = 0.0849$. Since $L < \boldsymbol{\kappa} = 0.2892$ and $R < \boldsymbol{\kappa} = 0.2892$, hence left and right kurtosis are flat relative to $\mathbf{b}\boldsymbol{\phi}(z)$ (see Figure 4).

Table 3: Numerical features of a non-symmetric mixture of two Cauchy densities.

M_1	M_2	m
-2.4955	2.4875	0.2606
$f_2(M_1; 2.5)$	$f_2(M_2; 2.5)$	$f_2(m; 2.5)$
0.2035	0.1270	0.0429

Example 3: This is about Alpha-Skew-normal density, given by Elal Olivero (2010),

$$f_3(x;\alpha) = \frac{1+(1-\alpha x)^2}{2+\alpha^2}\phi(x), \quad x \in \mathbb{R}, \quad \alpha \in \mathbb{R}.$$

For $\alpha = 3$, this density is bimodal with features in Table 4.
For this density $L = 0.2128$ and $R = 0.0952$. Since $L < \boldsymbol{\kappa} = 0.2892$ and $R < \boldsymbol{\kappa} = 0.2892$, therefore, left and right kurtosis are flat relative to $\mathbf{b}\boldsymbol{\phi}(z)$. (See Figure. 5)

Table 4: Numerical features of an Alpha-Skew-Normal density with $\alpha = 3$.

M_1	M_2	m
-1.2263	1.5399	0.3530
$f_3(M_1; 3)$	$f_3(M_2; 3)$	$f_3(m; 3)$
0.3914	0.1562	0.0341

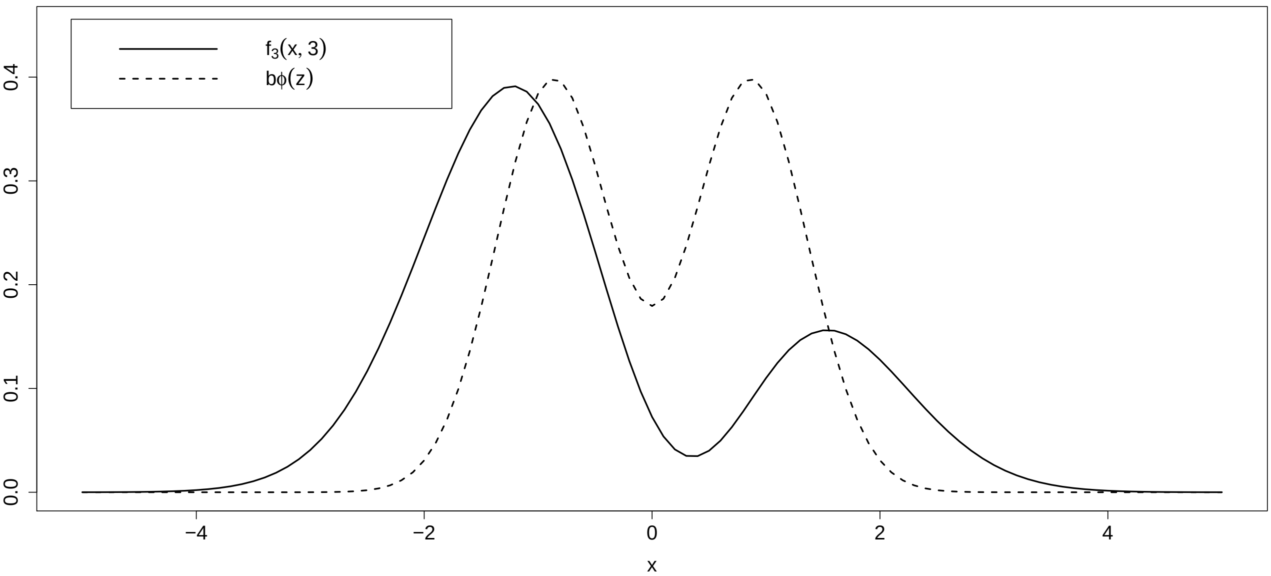

Figure 5: Graph of $f_3(x;3)$ and $\mathbf{b}\boldsymbol{\phi}(z)$.

Example 4: This is about location-scale Generalized Alpha-Skew-normal density, given by Sharafi et al. (2017),

$$f_4(x;\boldsymbol{\theta}) = \frac{(1-\alpha(\frac{x-\mu}{\sigma}))^2+1}{\sigma C(\alpha,\lambda)}\phi(\tfrac{x-\mu}{\sigma})\Phi(\lambda\tfrac{x-\mu}{\sigma}) \quad \alpha,\lambda,\mu\in\mathbb{R},\sigma>0, \tag{5.1}$$

where $\boldsymbol{\theta} = (\mu,\sigma,\alpha,\lambda)^T$, $C(\alpha,\lambda) = 1-\alpha b\delta+\frac{\alpha^2}{2}$, $b=\sqrt{\frac{2}{\pi}}$ and $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$. Now, we study this density for the following situations.

a: For $\boldsymbol{\theta} = \boldsymbol{\theta}_0 = (0, 0.8, 2, 0.01)$, this density is bimodal with features in Table 5.
For this density $L = 0.2906$ and $R = 0.1052$. Since $L > \boldsymbol{\kappa} = 0.2892$ and $R < \boldsymbol{\kappa} = 0.2892$, therefore, left kurtosis is peaked and right kurtosis is flat relative to $\mathbf{b}\boldsymbol{\phi}(z)$. (see Figure 6)

Table 5: Numerical features of a location-scale Generalized Alpha-Skew-Normal density with $\boldsymbol{\theta}_0 = (0, 0.8, 2, 0.01)$.

M_1	M_2	m
-0.8955	1.2443	0.4575
$f_4(M_1;\boldsymbol{\theta}_0)$	$f_4(M_2;\boldsymbol{\theta}_0)$	$f_4(m;\boldsymbol{\theta}_0)$
0.5085	0.1376	0.0727

b: For $\boldsymbol{\theta} = \boldsymbol{\theta}_1 = (0, 0.25, 2, 1)$, this density is bimodal with features in Table 6. For this density $L = 0.8406$ and $R = 0.9145$. Since $L > \boldsymbol{\kappa} = 0.2892$ and $R > \boldsymbol{\kappa} = 0.2892$, therefore, left and right kurtosis are peaked relative to $\mathbf{b}\boldsymbol{\phi}(z)$. (See Figure 7)

Table 6: Numerical features of a location-scale Generalized Alpha-Skew-Normal density with $\boldsymbol{\theta}_1 = (0, 0.25, 2, 1)$.

M_1	M_2	m
-0.1211	0.4028	0.1246
$f_4(M_1;\boldsymbol{\theta}_1)$	$f_4(M_2;\boldsymbol{\theta}_1)$	$f_4(m;\boldsymbol{\theta}_1)$
1.1611	1.3087	0.5202

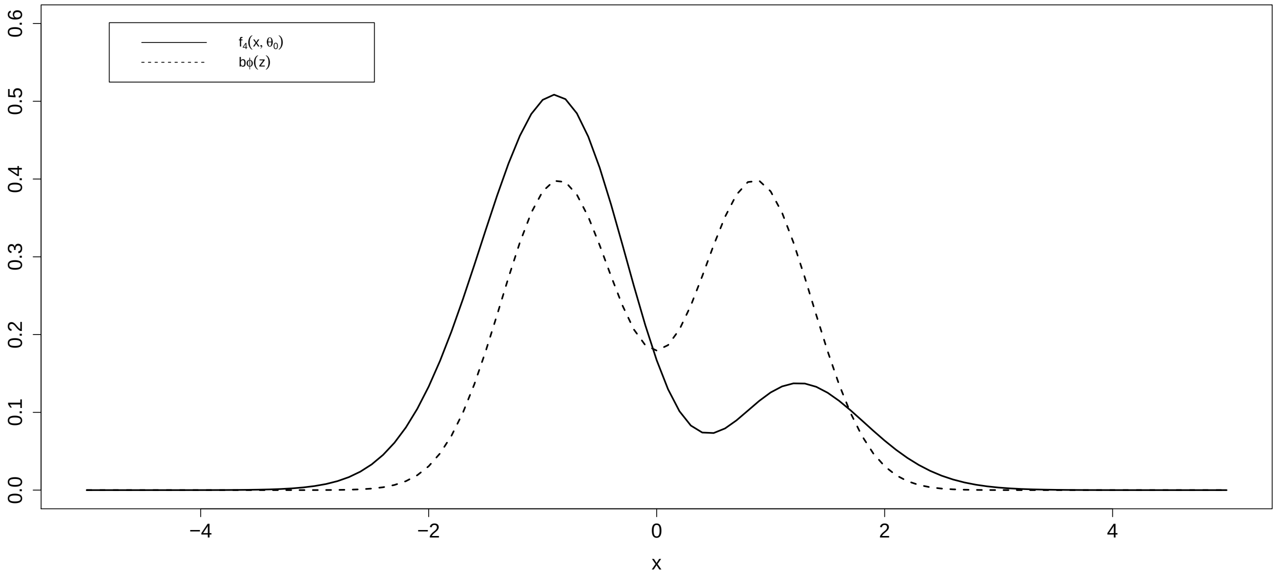

Figure 6: Graph of $f_4(x; \boldsymbol{\theta}_0)$ and $\mathbf{b}\boldsymbol{\phi}(z)$.

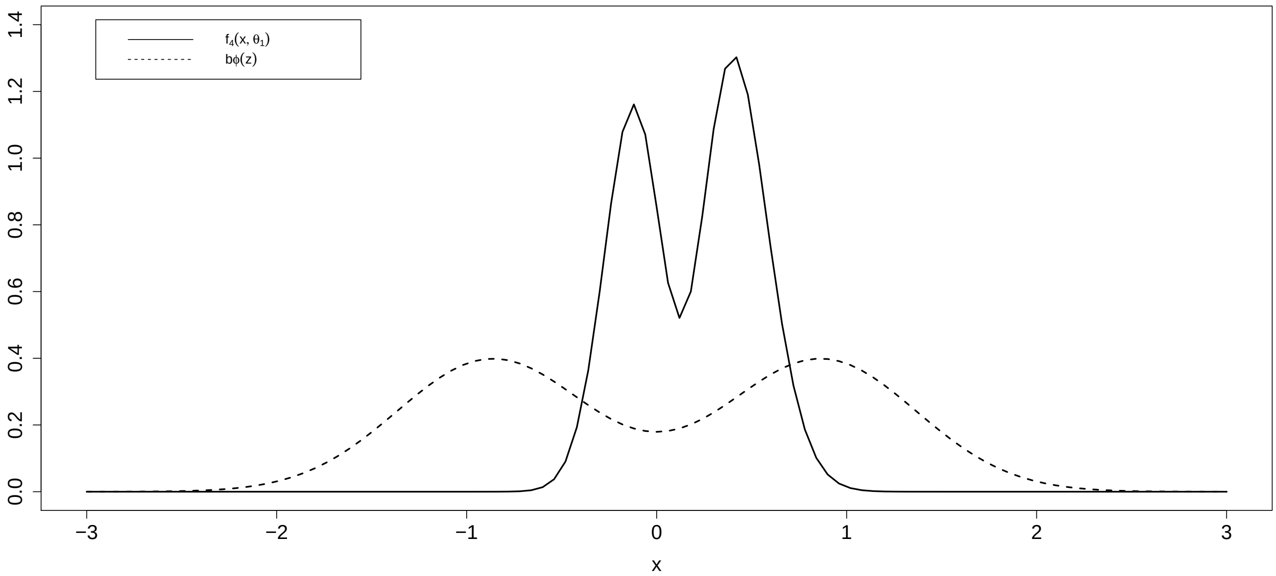

Figure 7: Graph of $f_4(x; \boldsymbol{\theta}_1)$ and $\mathbf{b}\boldsymbol{\phi}(z)$.

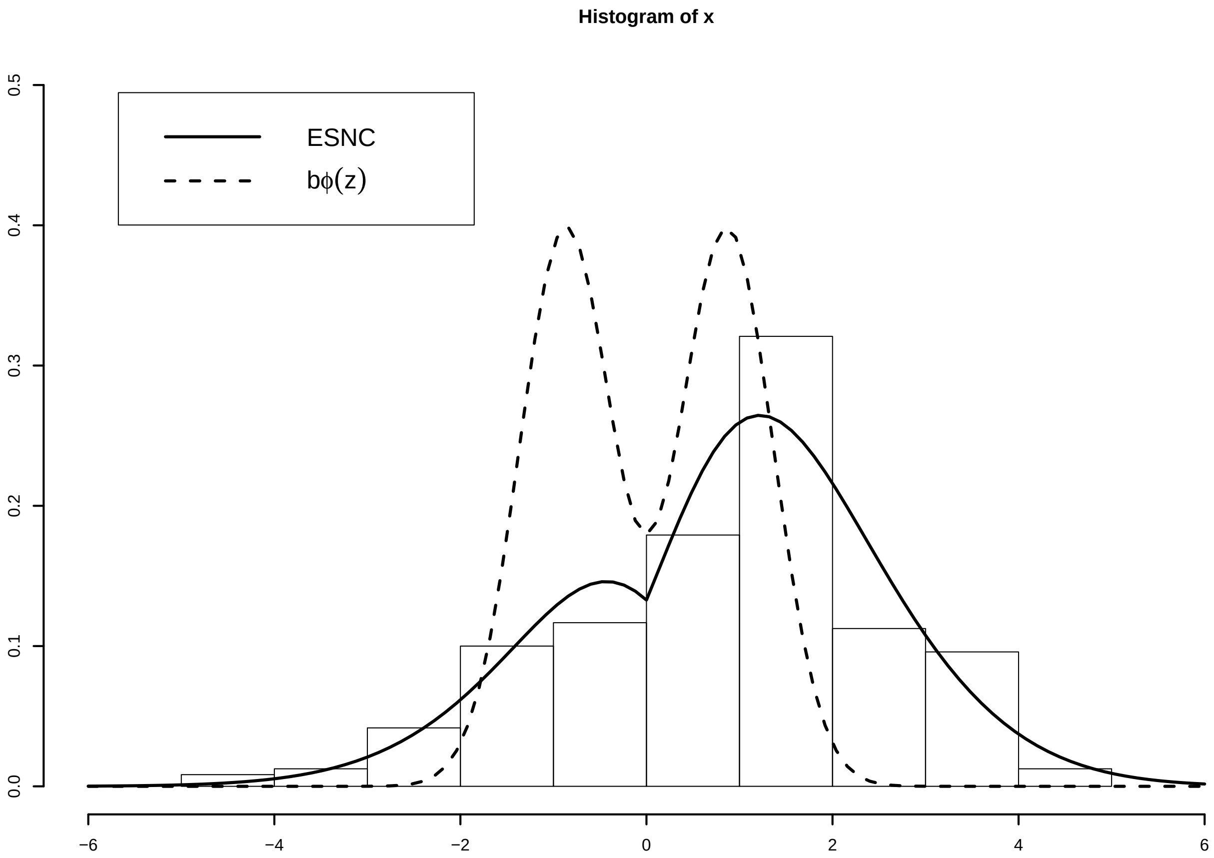

Figure 8: Histogram and fitted density of the real data and the density of $\mathbf{b}\boldsymbol{\phi}(z)$.

6. Application

To illustrate the application of kurtosis criteria for bimodal densities that have been introduced in this paper, we use a real data set. The set of data is the variable N-Cream available in the database Creaminess of cream cheese, which was studied by Arrue et al. (2015).

Arrue et al. (2015) introduced an extended skew-normal-Cauchy distribution with parameter $\theta = (\mu, \sigma, \alpha, \beta)$ (ESNC(θ)). One of the features of ESNC is uni-bimodality, which is controlled by parameter α. When $\alpha > 1$, the density is bimodal and is unimodal, if $\alpha < 1$.

Arrue et al. (2015) showed that ESNC($\hat{\theta}$) with $\hat{\theta} = (6.717, 1.781, 1.8631, 1.062)$ is better fitted on the data set. Since $\hat{\alpha} = 1.86 > 1$, the data are bimodal.

To calculate the left and right kurtosis values of the data, first, the data is centered by subtracting the $\hat{\mu} = 6.717$. Then by some calculation, we obtain $L = 0.1394275$ and $R = 0.1986749$. Since L and R are smaller than $\boldsymbol{\kappa} = 0.2892$, therefore, left and right kurtosis of the data are flat relative to $\mathbf{b}\boldsymbol{\phi}(z)$. Figure 8 indicates this conclusion.

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