(\(\alpha, \beta\))-Metrics with Killing \(\beta\) of Constant Length

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ABSTRACT: The class of (\(\alpha, \beta\))-metrics is a rich and important class of Finsler metrics, which is extensively studied. Here, we study (\(\alpha, \beta\))-metrics with Killing of constant length 1-form \(\beta\) and find a simplified formula for their Ricci curvatures. Then, we show that if \(F = \alpha + a\beta + b\frac{\beta^2}{\alpha}\) is an Einstein Finsler metric, then \(\alpha\) is an Einstein Riemann metric.

1. Introduction

The study of Killing fields of constant length is natural in some geometric constructions such as K-contact and Sasakian structure. There is a Killing field on the unit tangent bundle of a homogenous nonsymmetric Sasakian manifold with unit sectional curvature [3]. The existence of these fields on a Riemannian or more generally a Finsler manifold \(M\) cause some topological and geometrical restrictions on the manifold. S. Basco, X. Cheng and Z. Shen prove that a Finsler metric \(F = \alpha \pm \frac{\beta^2}{\alpha} + \epsilon\beta\) has vanishing \(S\)-curvature if and only if \(\beta\) is a Killing 1-form and with constant length with respect to the Riemannian metric \(\alpha\) [5]. In [6], the authors study some Einstein (\(\alpha, \beta\))-metrics with constant Killing 1-form \(\beta\). This motivates us to study (\(\alpha, \beta\))-metrics with Killing 1-form with constant length \(\beta\).

An (\(\alpha, \beta\))-metric on a manifold \(M\) is a Finsler metric \(F\) on \(M\) defined by \(F = f(\alpha, \beta)\), where \(\alpha = \sqrt{a_{ij}y^iy^j}\) is a Riemannian metric and \(\beta = b_iy^i\) is a 1-form on the manifold \(M\). Randers metrics are special (\(\alpha, \beta\))-metrics defined by \(F = \alpha + \beta\). Randers metrics have important applications both in mathematics and physics [5]. Let \(F = f(\alpha, \beta)\) be an (\(\alpha, \beta\))-metrics. Let \(\nabla\beta = b_{iji}dx^i \otimes dx^j\) denote the covariant derivative of \(\beta\) with respect to \(\alpha\). Put

\[
\begin{align*}
    r_{ij} &:= \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}), \\
    r_j &:= b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad r_{i0} := r_{ij} y^j, \quad s_{i0} := s_{ij} y^j, \quad r_0 := r_{ij} y^i, \quad s_0 := s_{ij} y^i.
\end{align*}
\]

Then the spray coefficients of \(F\) and \(\alpha\) are related by the following

\[
G^i = \bar{G}^i + B^i, \tag{1.1}
\]

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where
\[
B^i = \frac{E}{\alpha} y^i + \frac{\alpha f_{\beta}}{f_\alpha} s^i_0 - \frac{\alpha f_{\alpha \alpha}}{f_\alpha} (C + \frac{\alpha r_{\alpha \alpha}}{2\beta})(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta}),
\]  
(1.2)
\[
E = \frac{\beta f_{\beta}}{f}(\beta C + \alpha r_{\alpha \alpha}), \quad C = \frac{\alpha \beta}{2} (r_{\alpha \alpha} f_\alpha - 2a s_0 f_\beta)(\beta^2 f_\alpha + \alpha^2 f_{\alpha \alpha}),
\]  
(1.3)
\[
\gamma^2 = b^2 \alpha^2 - \beta^2, \quad b^2 = b b_i.
\]  
(1.4)
Based on well-known Berwald's formula, we have the following relation between Riemannian curvatures of $F$ and $\alpha$
\[
R^i_k = \bar{R}^i_k + \{2B^j B^i_{j,k} - y^i B^j_{j,k} - B^j_j B^i_{j,k} - B^j_k B^i_{j,k} + 2B^j_{k,i}\},
\]  
(1.5)
where “$|$” and “.” denote horizontal covariant derivative with respect to $\alpha$ and vertical derivative, respectively. Therefore, we have
\[
\bar{R}^i_1 = R^i_1 + \{2B^j B^i_{j,i} - B^j_j B^i_{j,i} + 2B^j_{i,j}\}.
\]  
(1.6)
Now, suppose that $s_i = 0$ and $r_{ij} = 0$. Then, we have the following.

**Theorem 1.1.** Let $F = f(\alpha, \beta)$ be an $(\alpha, \beta)$-metric with Killing of constant length 1-form $\beta$. Then the following holds
\[
R^i_1 = \bar{R}^i_1 + Q^i_1,
\]  
(1.7)
where $Q^i_1 := 2r s^i_0 + r^2 s^i s_{ij} + \frac{\beta}{\alpha}(f f_{\beta} f_{\alpha \beta} + f_{\alpha} f_{\beta} - f f_{\beta} f_{\alpha}) s^i_0 s_{00}$ and $r = \frac{\alpha f_{\beta}}{f_\alpha}$. Moreover, $F$ is Ricci-quadratic if and only if $Q^i_1$ is quadratic.

As an application of Theorem 1.1, we deal with a class of $(\alpha, \beta)$-metric given by $F = \alpha + a \beta + b \beta^2$ where $a$ and $b$ are two constant. More precisely, we have the following.

**Theorem 1.2.** Suppose that $\beta$ is Killing 1-form with constant length with respect to $\alpha$. Let $F = \alpha + a \beta + b \beta^2$ be an Einstein metric, i.e.,
\[
F^2 Ric(x) = R^i_1.
\]  
(1.8)
Then $\alpha$ is an Einstein metric.

2. Proof of Theorem 1.1

If $\beta$ is a Killing of constant length 1-form, then (1.2) reduces to $B^i = rs^i_0$ where $r = \frac{\alpha f_{\beta}}{f_\alpha}$. To prove Theorem 1.1, by (1.6), we need the following lemmas.

**Lemma 2.1.** $B^j B^i_{j,i} = 0$.

**Proof.** We have
\[
B^j B^i_{j,i} = rs^i_0 (r y^i s^j_0 + r s^j_0 y^i) = rs^i_0 (r y^i s^j_0 + r y^i s^j_0 + r y^i s^j_0 + r (s^j_0 y^i))
\]  
(2.1)
Using the relations $\alpha s^i_0 = 0$ and $b s^i_0 = 0$, we get first and second terms of the right hand side of (2.1) as follows
\[
rr y^i s^j_0 s^i_0 = \frac{r (f_{\beta} f_\alpha + f_{\alpha} f_\beta)}{\alpha f_\alpha^2} s_{00} s^i_0 = r As^i_0 s_{00}
\]  
(2.2)
and
\[
rr y^i s^j_0 s^i_0 = r As^i_0 s_{00}
\]  
(2.3)
where $A = \frac{f_a f_\alpha + f_\alpha a}{a f_\alpha}$. Plugging (2.2) and (2.3) into (2.1) and using $s_0 = -s_0$, we get the result.

**Lemma 2.2.** $y^j B_{i,j}^i = 0$.

**Proof.** We have

\begin{equation}
B_{i,j}^i = (r_j s_0^i + r s_{0ij})_i = r_j s_0^i + r_i s_k^i + r_j s_{0ij} + s_{kij}.
\end{equation}

Using the relation $s_0^i = 0$, we get

\begin{equation}
y^j B_{i,j}^i = y^j r_{ij} s_0^i + y^j r_{s_0} s_{0ij}.
\end{equation}

Taking into account $\alpha_{ij} = 0$, it is easy to see that

\begin{equation}
r_{ij} = \frac{\alpha f_\alpha f_{\beta\beta} - \alpha f_{\beta} f_{\alpha\beta}}{f_\alpha^2} \beta_{ij}.
\end{equation}

On the other hand, we have $\alpha f_\alpha + \beta f_{\beta\beta} = 0$. Then $\alpha f_{\alpha\beta} = -\beta f_{\beta\beta}$.

\begin{equation}
r_{ij} = \frac{(\alpha f_\alpha + \beta f_{\beta\beta}) f_{\beta\beta}}{f_\alpha^2} \beta_{ij} = \frac{f_{\beta\beta}}{f_\alpha^2} \beta_{ij}.
\end{equation}

Since by assumption $r_{ij} = 0$, we have $s_{ij} = b_{ij}$. Hence, we have

\begin{equation}
\beta_{ij} = (b_k y^k)^{ij} = b_{kj} y^k = s_{kj} y^k = s_{0j}.
\end{equation}

By taking vertical derivative from (2.6) with respect to $y^i$ implies that

\begin{equation}
r_{ij,i} = \left( \frac{f_{\beta\beta}}{f_\alpha^2} \right) s_{0j} + \frac{f_{\beta\beta}}{f_\alpha^2} s_{ij}.
\end{equation}

Contracting (2.7) with $y^j s_0^i$ yields the first term of the left hand side of (2.5), i.e.,

\begin{equation}
y^j r_{ij,i} s_0^i = \left( \frac{f_{\beta\beta}}{f_\alpha^2} \right) s_{0j} y^j + \frac{f_{\beta\beta}}{f_\alpha^2} s_{ij} y^j = \frac{f_{\beta\beta}}{f_\alpha^2} s_{0j} s_{0i},
\end{equation}

in which we have used the fact $s_{0j} y^j = s_{0j} = 0$.

Now we are going to find the second term of the left hand side of (2.5). First, note that the following relation holds

\begin{equation}
b_{i} s_{0ij} + b_{ij} s_{0i} = 0.
\end{equation}

Using (2.9) and by direct computation, we have

\begin{equation}
r_{i} s_{0ij} = \alpha f_\alpha f_{\beta\beta} s_{0ij} + \frac{\alpha f_{\alpha\beta}}{f_\alpha^2} f_{\beta\beta} s_{ij}s_{0i} - \frac{\alpha f_{\beta\alpha}}{f_\alpha^2} f_{\beta\beta} s_{ij}s_{0i}.
\end{equation}

Contracting (2.10) with $y^i$ and using $\alpha f_{\alpha\beta} = -\beta f_{\beta\beta}$ and $\alpha f_{\beta\beta} y^i = 0$, we get

\begin{equation}
y^i r_{i} s_{0ij} = -\frac{f_{\beta\beta}}{f_\alpha^2} s_{0i} s_{0i}.
\end{equation}

Substituting (2.8) and (2.11) into (2.5), we get the result.

**Lemma 2.3.** $B_{i,j}^i B_{i,j}^i = -2r A s_{0i} s_{0j} - r^2 s_{ij} s_{ij}$, where $A = \frac{f_a f_\alpha + f_\alpha a}{aj^2}$.

**Proof.**

\begin{equation}
B_{i,j}^i B_{i,j}^i = (r_j s_0^i + r s_{0ij})(r_j s_0^i + r s_{0ij})
= r_j r s_0^i s_0^i + r r s_{0i} s_{0i} + r_j r s_{0ij} s_{0ij} + r^2 s_{ij} s_{ij}.
\end{equation}
Using the relation $\alpha_j s^i_j = \frac{1}{2} s_{0i}$ and a direct computation, we get
\[ r_j s^i_0 s^j_i = r_i s^i_0 s^j_i = -As^i_0 s_{0i}. \]
(2.13)

Using the relations $\alpha_i s^i_0 = 0$ and $b_i s^i_0 = 0$, we get
\[ r_j r_i s^i_0 s^j_0 = 0. \]
(2.14)

Plugging (2.13) and (2.14) into (2.12), we get the result.

**Corollary 2.5.** A Randers metric $F = \alpha + \beta$ with Killing 1-form with constant length is Ricci-quadratic if and only if
\[ s^0_{0i} = 0. \]
(2.17)

Let us remark that in Bao-Shen’s sphere Randers metric $F = \alpha + \beta$, the 1-form $\beta$ is a Killing 1-form with constant length [6]. It is proved that $F$ has non-zero constant flag curvature. Hence, it is an Einstein Randers metric. We know that if an Einstein Randers metric is Ricci-quadratic, then it is either Riemannian metric or Ricci-flat. Hence, Bao-Shen’s sphere Randers metric is not Ricci-quadratic, because it is neither Riemannian metric nor Ricci-flat.

**Example 1.** The projective spherical metric on $R^3$ is given by the following:
\[ \alpha := \frac{\sqrt{(1 + ||X||^2)||Y||^2 - <X,Y>^2}}{1+<X,Y>}, \quad X \in R^3, \quad Y \in T_X R^3 \]
where $<,>$ and $||.||$ denote the Euclidean inner product and norm on $R^3$, respectively. Put $X = (x,y,z)$ and $Y = (u,v,w)$. Suppose that $\beta = b_1 u + b_2 v + b_3 w$ is a Killing 1-form of $\alpha$. It is proved that
\[ b_1 = \frac{1}{1+<X,Y>} (Q^1 u + Q^2 z + C^1), \]
\[ b_2 = \frac{1}{1+<X,Y>} (Q^1 z + Q^2 z + C^2), \]
\[ b_3 = \frac{1}{1+<X,Y>} (Q^1 x + Q^1 y + C^3), \]
where $Q = (Q^i_j)$ is an antisymmetric real matrix and $C = (C^i)$ is a constant vector in $R^3$. Let $C = (0,1,0)$ and $Q^1_1 = 0, Q^1_3 = 1$. Using a Maple program shows that in this case, $\beta$ is a Killing 1-form with unit length with respect to $\alpha$, which is not a closed 1-form.
3. Proof of Theorem 1.2

In this section, as an application of Theorem 1.1, we deal with the metric $F = \alpha + a\beta + b\frac{x^2}{\alpha}$, where $a$ and $b$ are two non-zero real constant. This class of Finsler metrics contains the class of Randers metrics as a special case.

Plugging (1.8) into (1.7), led to the following equation

$$ \text{Rat} + 2a\alpha \text{Irrat} = 0, \tag{3.1} $$

where

$$ \text{Rat} := (a^2 s^{ij} s_{ij} - \text{Ric})\alpha^6 + (b R_{\beta}\beta^2 + \bar{T}_i - a^2 \text{Ric} \beta^2 $$

$$ + 4b^2 s^{ij} s_{ij} \beta^2 + 2a^2 s^2_{ij} s_{ij} - 4bs^2 s_{ij} \beta^2 + 4bs^2 s_{ij} \beta^2 $$

$$ + 6a^2 bs^2 s_{ij} \beta^2 - 8b^2 s^2 s_{ij} \beta^2 + 2b^2 \text{Ric} \beta^4 $$

$$ + 3a^2 b R_{\beta} \beta^4 + (3b^2 \bar{T}_i \beta^4 + 4b^3 s^2_{ij} \beta^5 - 3a^2 b^2 \text{Ric} \beta^6 - 2b^3 \text{Ric} \beta^6 $$

$$ - 12b^3 s^2_{ij} s_{ij} \beta^4)\alpha^4 + (-b^4 \text{Ric} \beta^8 + a^2 b^4 \text{Ric} \beta^8 - b^3 \bar{T}_i \beta^6)\alpha^2 $$

$$ + b^5 \text{Ric} \beta^{10}, \tag{3.2} $$

and

$$ \text{Irrat} := (s^2_{ij} - 2bs^2 s_{ij} \beta - \text{Ric} \beta)\alpha^8 + (2b R_{\beta} \beta^3 - 2b^2 s^2 s_{ij} \beta^2 - 2bs^2 s_{ij} \beta^2)\alpha^6 $$

$$ + (-8b^2 s^2 s_{ij} \beta^3 + b^2 s^2 s_{ij} \beta^4)\alpha^4 - 2b^3 \text{Ric} \beta^7 \alpha^2 + b^4 \text{Ric} \beta^9. \tag{3.3} $$

It is easy to see that $F$ is Einstein metric if and only if $\text{Rat} = 0$ and $\text{Irrat} = 0$. We can rewrite $\text{Rat} = 0$, as follows:

$$ \mu \alpha^2 + b^5 \text{Ric} \beta^{10} = 0, \tag{3.4} $$

where

$$ \mu := (a^2 s^{ij} s_{ij} - \text{Ric})\alpha^8 + (b R_{\beta}\beta^2 + \bar{T}_i - a^2 \text{Ric} \beta^2 $$

$$ + 4b^2 s^{ij} s_{ij} \beta^2 + 2a^2 s^2_{ij} s_{ij} - 4bs^2 s_{ij} \beta^2 + 4bs^2 s_{ij} \beta^2 $$

$$ + 6a^2 bs^2 s_{ij} \beta^2 - 8b^2 s^2 s_{ij} \beta^2 + 2b^2 \text{Ric} \beta^4 $$

$$ + 3a^2 b R_{\beta} \beta^4 + (3b^2 \bar{T}_i \beta^4 + 4b^3 s^2_{ij} \beta^5 - 3a^2 b^2 \text{Ric} \beta^6 - 2b^3 \text{Ric} \beta^6 $$

$$ - 12b^3 s^2_{ij} s_{ij} \beta^4)\alpha^4 - b^4 \text{Ric} \beta^8 + a^2 b^3 \text{Ric} \beta^8 - b^3 \bar{T}_i \beta^6 \tag{3.5} $$

It means that $\alpha^2$ divides $\text{Ric}(x)$, which is impossible unless $\text{Ric}(x) = 0$ or $\beta = 0$. In each case, we must have $\mu = 0$. If $\beta = 0$, then the proof is done. Suppose that $F$ is Ricci-flat, i.e., $\text{Ric} = 0$. Then the equation $\mu = 0$ is reduced to the following

$$ \lambda \alpha^2 - b^5 \bar{T}_i \beta^6 = 0, \tag{3.6} $$

where

$$ \lambda := a^2 s^{ij} s_{ij} \alpha^6 + (\bar{T}_i + 4b^2 s^{ij} s_{ij} \beta^2 + 2a^2 s^2_{ij} s_{ij} - 4bs^2 s_{ij} \beta^2 + 4bs^2 s_{ij} \beta^2 $$

$$ - 4bs^2 s_{ij} \beta^2)\alpha^4 - (-3b \bar{T}_i \beta^2 - 4b^3 s^2 s_{ij} \beta^2 - 6a^2 bs^2 s_{ij} \beta^2 - 8b^2 s^2 s_{ij} \beta^2 $$

$$ + 3b^2 \bar{T}_i \beta^2 + 4b^3 s^2 s_{ij} \beta^5 - 12b^3 s^2 s_{ij} \beta^4 \tag{3.7} $$

Equation (3.6) implies that $\alpha^2$ divides $\bar{R}_i$, i.e., $\bar{R}_i = c\alpha^2$ where $c$ is a constant by Riemannian Schur lemma. This means that $\alpha$ is an Einstein metric. This completes the proof.

Q.E.D.

**Remark 3.1.** Let $\alpha$ be the projective spherical metric on $R^3$ and $\beta = \lambda(zdx + dy - xdz)$, where $\lambda = \frac{1}{|x|^2}$. The Riemannian metric $\alpha$ is of constant curvature $K = 1$. Thus $\alpha$ is Einstein metric and $\bar{R}_i = 2\alpha^2$. A direct computation, using a Maple program shows that $F = \alpha + a\beta + b\frac{x^2}{\alpha}$ is not Einstein metric for any $a$ and $b.$
Therefore, the converse of Theorem 1.2 is not true.

4. Appendix

Maple program of converse of theorem 1.2

\[
\alpha := \sqrt{(x^2+y^2+z^2+1)(u^2+v^2+w^2)-(u*x+v*y+w*z)^2)/(x^2+y^2+z^2+1)}:
\]

\[
\begin{align*}
    a_{11} &= \text{simplify}\left(\text{diff}(\alpha^2/2,u,u)\right) \\
    a_{12} &= \text{simplify}\left(\text{diff}(\alpha^2/2,u,v)\right) \\
    a_{13} &= \text{simplify}\left(\text{diff}(\alpha^2/2,u,w)\right) \\
    a_{21} &= a_{12} \\
    a_{22} &= \text{simplify}\left(\text{diff}(\alpha^2/2,v,v)\right) \\
    a_{23} &= \text{simplify}\left(\text{diff}(\alpha^2/2,v,w)\right) \\
    a_{31} &= a_{13} \\
    a_{32} &= a_{23} \\
    a_{33} &= \text{simplify}\left(\text{diff}(\alpha^2/2,w,w)\right) \\
    C_1 &= 0; -1; C_2 := 1; -1; C_3 := 0; -1; Q_1,3 := 1; -1; Q_1,2 := 0; -1; Q_2,3 := 0; \\
    b_1 &= Q_1,2*y+Q_1,3*z+C_1+(C_1*x+C_2*y+C_3*z)*x \\
    b_2 &= -Q_1,2*x+Q_2,3*z+C_2+(C_1*x+C_2*y+C_3*z)*y \\
    b_3 &= -Q_1,3*x-Q_2,3*y+C_3+(C_1*x+C_2*y+C_3*z)*z \\
    b_1 &= \text{simplify}(a_{11}*b_1+a_{12}*b_2+a_{13}*b_3) \\
    b_2 &= \text{simplify}(a_{21}*b_1+a_{22}*b_2+a_{23}*b_3) \\
    b_3 &= \text{simplify}(a_{31}*b_1+a_{32}*b_2+a_{33}*b_3) \\
    \beta &= b_1*u+b_2*v+b_3*w
\end{align*}
\]

In this part we want to find \( r_{ij} \) and \( s_{ij} \):

\[
\begin{align*}
    a_{ij} &= \text{Matrix}([[a_{11},a_{12},a_{13}],[a_{21},a_{22},a_{23}],[a_{31},a_{32},a_{33}]]): \\
    R_1 &= \text{LinearAlgebra:-MatrixInverse}(a_{ij}): \\
    a_{11} &= R_1[1,1]: \\
    a_{12} &= R_1[1,2]: \\
    a_{13} &= R_1[1,3]: \\
    a_{21} &= R_1[2,1]: \\
    a_{22} &= R_1[2,2]: \\
    a_{23} &= R_1[2,3]: \\
    a_{31} &= R_1[3,1]: \\
    a_{32} &= R_1[3,2]: \\
    a_{33} &= R_1[3,3]: \\
    b_1 &= \text{simplify}(a_{11}*b_1+a_{12}*b_2+a_{13}*b_3): \\
    b_2 &= \text{simplify}(a_{21}*b_1+a_{22}*b_2+a_{23}*b_3): \\
    b_3 &= \text{simplify}(a_{31}*b_1+a_{32}*b_2+a_{33}*b_3): \\
    B_2 &= \text{simplify}(b_1*b_1+b_2*b_2+b_3*b_3): \\
    \end{align*}
\]

Here, we compute the Christoffel symbols of the Riemannian metric “alpha:
In this part we will compute the geodesic sprays and projective quantities of the alpha:

\[ G_1_a := \frac{1}{2} \text{simplify}(G_1_1 u^2 + G_1_2 uv + G_1_3 uw + G_1_2 vu + G_1_2 v^2 + G_1_2 vw + G_1_3 wu + G_1_3 wv + G_1_3 w^2) \]

\[ G_2_a := \frac{1}{2} \text{simplify}(G_2_1 u^2 + G_2_2 uv + G_2_3 uw + G_2_2 vu + G_2_2 v^2 + G_2_2 vw + G_2_3 wu + G_2_3 wv + G_2_3 w^2) \]

\[ G_3_a := \frac{1}{2} \text{simplify}(G_3_1 u^2 + G_3_2 uv + G_3_3 uw + G_3_2 vu + G_3_2 v^2 + G_3_2 vw + G_3_3 wu + G_3_3 wv + G_3_3 w^2) \]

Here, we have \( \alpha \) is projectively flat. Therefore, \( \alpha \) is of constant curvature by Beltrami theorem.

Every Riemannian metric with constant curvature is Einstein metric. Thus \( \alpha \) is Einstein metric.

\[ P := G_1_a / u \]

\[ X_i := \text{simplify}(P^2 - (\text{diff}(P, x)) u - (\text{diff}(P, y)) v - (\text{diff}(P, z)) w) \]

\[ K := \text{simplify}(X_i / \alpha^2) \]

This means that \( \alpha \) is of constant flag curvature \( K = 1 \). Hence, we have \( R^i_\ell = 2 \alpha^2 \).

Now we are going to compute \( b_{i-j} = \text{diff}(b_i, x_j) - G_{kij} b_k \):

\[ b_{11} := \text{simplify}(\text{diff}(b_1, x) - G_{1111} b_1 - G_{2111} b_2 - G_{3111} b_3) \]
\[ b_{12} := \text{simplify} \left( \frac{\partial b_1}{\partial y} - G_{12} b_1 - G_{21} b_2 - G_{31} b_3 \right) \]
\[ b_{13} := \text{simplify} \left( \frac{\partial b_1}{\partial z} - G_{13} b_1 - G_{23} b_2 - G_{33} b_3 \right) \]
\[ b_{21} := \text{simplify} \left( \frac{\partial b_2}{\partial x} - G_{21} b_1 - G_{12} b_2 - G_{32} b_3 \right) \]
\[ b_{22} := \text{simplify} \left( \frac{\partial b_2}{\partial y} - G_{22} b_1 - G_{12} b_2 - G_{32} b_3 \right) \]
\[ b_{23} := \text{simplify} \left( \frac{\partial b_2}{\partial z} - G_{23} b_1 - G_{13} b_2 - G_{33} b_3 \right) \]
\[ b_{31} := \text{simplify} \left( \frac{\partial b_3}{\partial x} - G_{31} b_1 - G_{13} b_2 - G_{31} b_3 \right) \]
\[ b_{32} := \text{simplify} \left( \frac{\partial b_3}{\partial y} - G_{32} b_1 - G_{13} b_2 - G_{32} b_3 \right) \]
\[ b_{33} := \text{simplify} \left( \frac{\partial b_3}{\partial z} - G_{33} b_1 - G_{13} b_2 - G_{33} b_3 \right) \]

We know that
\[ s_{ij} := \frac{1}{2} (b_{ij} - b_{ji}) \]
\[ s_{11} := 0; \quad s_{22} := 0; \quad s_{33} := 0; \]
\[ s_{12} := \text{simplify} \left( \frac{b_{12} - b_{21}}{2} \right); \quad s_{21} := -s_{12}; \quad \text{simplify} \left( 2s_{12} - \text{Closeness}_{12} \right) \]
\[ s_{13} := \text{simplify} \left( \frac{b_{13} - b_{31}}{2} \right); \quad s_{31} := -s_{13}; \quad \text{simplify} \left( 2s_{13} - \text{Closeness}_{13} \right) \]
\[ s_{23} := \text{simplify} \left( \frac{b_{23} - b_{32}}{2} \right); \quad s_{32} := -s_{23}; \quad \text{simplify} \left( 2s_{23} - \text{Closeness}_{23} \right) \]
\[ s_{10} := \text{simplify} \left( s_{11} u + s_{12} v + s_{13} w \right) \]
\[ s_{20} := \text{simplify} \left( s_{21} u + s_{22} v + s_{23} w \right) \]
\[ s_{30} := \text{simplify} \left( s_{31} u + s_{32} v + s_{33} w \right) \]

Here, we compute
\[ s_{10} := \text{simplify} \left( a_{11} s_{11} + a_{12} s_{21} + a_{13} s_{31} \right) \]
\[ s_{12} := \text{simplify} \left( a_{21} s_{11} + a_{22} s_{22} + a_{23} s_{32} \right) \]
\[ s_{13} := \text{simplify} \left( a_{31} s_{11} + a_{32} s_{21} + a_{33} s_{31} \right) \]
\[ s_{20} := \text{simplify} \left( a_{11} s_{12} + a_{12} s_{22} + a_{13} s_{32} \right) \]
\[ s_{22} := \text{simplify} \left( a_{22} s_{22} + a_{23} s_{32} + a_{32} s_{32} \right) \]
\[ s_{23} := \text{simplify} \left( a_{31} s_{12} + a_{32} s_{22} + a_{33} s_{32} \right) \]
\[ s_{30} := \text{simplify} \left( a_{11} s_{13} + a_{12} s_{23} + a_{13} s_{33} \right) \]
\[ s_{32} := \text{simplify} \left( a_{21} s_{13} + a_{22} s_{23} + a_{23} s_{33} \right) \]
\[ s_{33} := \text{simplify} \left( a_{31} s_{13} + a_{32} s_{23} + a_{33} s_{33} \right) \]

Here, we compute
\[ s_{11} := \text{simplify} \left( a_{11} s_{11} + a_{12} s_{12} + a_{13} s_{13} \right) \]
\[ s_{12} := \text{simplify} \left( a_{21} s_{11} + a_{22} s_{12} + a_{23} s_{13} \right) \]
\[ s_{13} := \text{simplify} \left( a_{31} s_{11} + a_{32} s_{12} + a_{33} s_{13} \right) \]
\[ s_{21} := a_{11}s_{2_1}+a_{12}s_{2_2}+a_{13}s_{2_3}; \]
\[ s_{22} := a_{21}s_{2_1}+a_{22}s_{2_2}+a_{23}s_{2_3}; \]
\[ s_{23} := a_{31}s_{2_1}+a_{32}s_{2_2}+a_{33}s_{2_3}; \]
\[ s_{31} := a_{11}s_{3_1}+a_{12}s_{3_2}+a_{13}s_{3_3}; \]
\[ s_{32} := a_{21}s_{3_1}+a_{22}s_{3_2}+a_{23}s_{3_3}; \]
\[ s_{33} := a_{31}s_{3_1}+a_{32}s_{3_2}+a_{33}s_{3_3}; \]

Here, we compute \( S_{ij}S_{ij} \):

\[ S_{ij}S_{ij} := \text{simplify}(s_{11}s_{11}+s_{12}s_{12}+s_{13}s_{13}+s_{21}s_{21}+s_{22}s_{22}+s_{23}s_{23}+s_{31}s_{31}+s_{32}s_{32}+s_{33}s_{33}); \]
R^i_i := simplify((2*alpha^2+Q)/F^2):

If R^i_i is a function of only (x,y,z), then F is an Einstein metric. But this is not the case. Hence, F is not Einstein metric!!

References