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# $(\alpha, \beta)$-Metrics with Killing $\beta$ of Constant Length

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**ABSTRACT:** The class of $(\alpha, \beta)$-metrics is a rich and important class of Finsler metrics, which is extensively studied. Here, we study $(\alpha, \beta)$-metrics with Killing of constant length 1-form $\beta$ and find a simplified formula for their Ricci curvatures. Then, we show that if $F = \alpha + a\beta + b\,\frac{\beta^2}{\alpha}$ is an Einstein Finsler metric, then $\alpha$ is an Einstein Riemann metric.

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## 1. Introduction

The study of Killing fields of constant length is natural in some geometric constructions such as K-contact and Sasakian structure. There is a Killing field on the unit tangent bundle of a homogenous nonsymmetric Sasakian manifold with unit sectional curvature [3]. The existence of these fields on a Riemannian or more generally a Finsler manifold $M$ cause some topological and geometrical restrictions on the manifold. S. Basco, X. Cheng and Z. Shen prove that a Finsler metric $F = \alpha \pm \frac{\beta^2}{\alpha} + \epsilon\beta$ has vanishing $S$-curvature if and only if $\beta$ is a Killing 1-form and with constant length with respect to the Riemannian metric $\alpha$ [5]. In [6], the authors study some Einstein $(\alpha, \beta)$-metrics with constant Killing 1-form $\beta$. This motivates us to study $(\alpha, \beta)$-metrics with Killing 1-form with constant length $\beta$.

An $(\alpha, \beta)$-metric on a manifold $M$ is a Finsler metric $F$ on $M$ defined by $F = f(\alpha, \beta)$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_iy^i$ is a 1-form on the manifold $M$. Randers metrics are special $(\alpha, \beta)$-metrics defined by $F = \alpha + \beta$. Randers metrics have important applications both in mathematics and physics [5]. Let $F = f(\alpha, \beta)$ be an $(\alpha, \beta)$-metrics. Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ denote the covariant derivative of $\beta$ with respect to $\alpha$. Put

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}) \ , \ \ s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j := b^ir_{ij}, \ s_j := b^is_{ij}, \ r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \ r_0 := r_jy^j, \ s_0 := s_jy^j.$$

Then the spray coefficients of $F$ and $\alpha$ are related by the following

$$G^i = \bar{G}^i + B^i, \tag{1.1}$$

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where

$$B^i = \frac{E}{\alpha}y^i + \frac{\alpha f_\beta}{f_\alpha}s^i_0 - \frac{\alpha f_{\alpha\alpha}}{f_\alpha}(C + \frac{\alpha r_{oo}}{2\beta})(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta}), \tag{1.2}$$

$$E = \frac{\beta f_\beta}{f}(\beta C + \alpha r_{oo}), \;\; C = \frac{\alpha\beta}{2}(r_{00}f_\alpha - 2\alpha s_0 f_\beta)(\beta^2 f_\alpha + \alpha\gamma^2 f_{\alpha\alpha}), \tag{1.3}$$

$$\gamma^2 = b^2\alpha^2 - \beta^2, \;\; b^2 = b^i b_i. \tag{1.4}$$

Based on well-known Berwald's formula, we have the following relation between Riemannian curvatures of $F$ and $\alpha$

$$R^i_k = \bar{R}^i_k + \{2B^j B^i_{\;.j.k} - y^j B^i_{\;|j.k} - B^i_{\;.j}B^j_{\;.k} + 2B^i_{\;|k}\}, \tag{1.5}$$

where "|" and "." denote horizontal covariant derivative with respect to $\alpha$ and vertical derivative, respectively. Therefore, we have

$$R^i_i = \bar{R}^i_i + \{2B^j B^i_{\;.j.i} - y^j B^i_{\;|j.i} - B^i_{\;.j}B^j_{\;.i} + 2B^i_{\;|i}\}. \tag{1.6}$$

Now, suppose that $s_i = 0$ and $r_{ij} = 0$. Then, we have the following.

**Theorem 1.1.** Let $F = f(\alpha, \beta)$ be an $(\alpha, \beta)$- metric with Killing of constant length 1-form $\beta$. Then the following holds

$$R^i_i = \bar{R}^i_{\;i} + Q^i_{\;i}, \tag{1.7}$$

where $Q^i_{\;i} := 2rs^i_{\;0|i} + r^2 s^{ij}s_{ij} + \frac{2}{f^3_\alpha}(ff_\beta f_{\alpha\beta} + f_\alpha f^2_\beta - ff_\alpha f_{\beta\beta})s^i_{\;0}s_{i0}$ and $r = \frac{\alpha f_\beta}{f_\alpha}$. Moreover, $F$ is Ricci-quadratic if and only if $Q^i_i$ is quadratic.

As an application of Theorem 1.1, we deal with a class of $(\alpha, \beta)$-metric given by $F = \alpha + a\beta + b\,\frac{\beta^2}{\alpha}$ where $a$ and $b$ are two constant. More precisely, we have the following.

**Theorem 1.2.** Suppose that $\beta$ is Killing 1-form with constant length with respect to $\alpha$. Let $F = \alpha + a\beta + b\,\frac{\beta^2}{\alpha}$ be an Einstein metric, i.e.,

$$F^2 Ric(x) = R^i_{\;i}. \tag{1.8}$$

Then $\alpha$ is an Einstein metric.

## 2. Proof of Theorem 1.1

If $\beta$ is a Killing of constant length 1-form, then (1.2) reduces to $B^i = rs^i_0$ where $r = \frac{\alpha f_\beta}{f_\alpha}$. To prove Theorem 1.1, by (1.6), we need the following lemmas.

**Lemma 2.1.** $B^j B^i_{\;.j.i} = 0$.

**Proof.** We have

$$\begin{aligned} B^j B^i_{\;.j.i} &= rs^j_0(r_{y^j}s^i_0 + rs^i_j)_{y^i} = rs^j_0(r_{y^j y^i}s^i_0 + r_{y^j}s^i_i + r_{y^i}s^i_j + r(s^i_j)_{y^i}) \\ &= rr_{y^j y^i}s^i_0 s^j_0 + r_{y^i}s^i_j rs^j_0 = rr_{y^j y^i}s^i_0 s^j_0 + rr_{y^i}s^j_0 s^i_j \end{aligned} \tag{2.1}$$

Using the relations $\alpha_i s^i_0 = 0$ and $b_i s^i_0 = 0$, we get first and second terms of the right hand side of (2.1) as follows

$$rr_{y^j y^i}s^i_0 s^j_0 = \frac{r(f_\beta f_\alpha + f_{\beta\alpha}f)}{\alpha {f_\alpha}^2}s_{i0}s^i_0 = rAs^i_0 s_{i0} \tag{2.2}$$

and

$$rr_{y^i}s^i_j s^j_0 = rAs_{0i}s^i_0 \tag{2.3}$$

where $A = \frac{f_\alpha f_\beta + f f_{\alpha\beta}}{\alpha f_\alpha^2}$. Plugging (2.2) and (2.3) into (2.1) and using $s_{0i} = -s_{i0}$, we get the result.

**Lemma 2.2.** $y^j B^i_{|j.i} = 0$.

**Proof.** We have

$$B^i_{|j.k} = (r_{|j} s^i_0 + r s^i_{0|j})_{.k} = r_{|j.k} s^i_0 + r_{|j} s^i_k + r_{.k} s^i_{0|j} + r s^i_{k|j}. \tag{2.4}$$

Using the relation $s^i_i = 0$, we get

$$y^j B^i_{|j.i} = y^j r_{|j.i} s^i_0 + y^j r_{.i} s^i_{0|j}. \tag{2.5}$$

Taking into account $\alpha_{|j} = 0$, it is easy to see that

$$r_{|j} = \frac{\alpha f_\alpha f_{\beta\beta} - \alpha f_\beta f_{\alpha\beta}}{f_\alpha^2} \beta_{|j}$$

On the other hand, we have $\alpha f_{\beta\alpha} + \beta f_{\beta\beta} = 0$. Thus $\alpha f_{\alpha\beta} = -\beta f_{\beta\beta}$.

$$r_{|j} = \frac{(\alpha f_\alpha + \beta f_\beta) f_{\beta\beta}}{f_\alpha^2} \beta_{|j} = \frac{f f_{\beta\beta}}{f_\alpha^2} \beta_{|j} \tag{2.6}$$

Since by assumption $r_{ij} = 0$, we have $s_{ij} = b_{i|j}$. Hence, we have

$$\beta_{|j} = (b_k y^k)_{|j} = b_{k|j} y^k = s_{kj} y^k = s_{0j}.$$

By taking vertical derivative from (2.6) with respect to $y^i$ implies that

$$r_{|j.i} = (\frac{f f_{\beta\beta}}{f_\alpha^2})_{.i} s_{0j} + \frac{f f_{\beta\beta}}{f_\alpha^2} s_{ij} \tag{2.7}$$

Contracting (2.7) with $y^j s^i_0$ yields the first term of the left hand side of (2.5), i.e.,

$$y^j r_{|j.i} s^i_0 = (\frac{f f_{\beta\beta}}{f_\alpha^2})_{.i} s^i_0 s_{0j} y^j + \frac{f f_{\beta\beta}}{f_\alpha^2} s^i_0 s_{ij} y^j = \frac{f f_{\beta\beta}}{f_\alpha^2} s^i_0 s_{i0}, \tag{2.8}$$

in which we have used the fact $s_{0j} y^j = s_{00} = 0$.
Now we are going to find the second term of the left hand side of (2.5). First, note that the following relation holds

$$b_i s^i_{0|j} + b_{i|j} s^i_0 = 0. \tag{2.9}$$

Using (2.9) and by direct computation, we have

$$\begin{aligned} r_{.i} s^i_{0|j} &= \alpha_{.i} \frac{f_\beta}{f_\alpha} s^i_{0|j} - \frac{\alpha}{f_\alpha} f_{\beta\beta} s_{ij} s^i_0 + \frac{\alpha}{f_\alpha} f_{\beta\alpha} \alpha_i s^i_{0|j} - \frac{\alpha f_\beta}{f_\alpha^2} f_{\alpha\alpha} \alpha_i s^i_{0|j} \\ &\quad + \frac{\alpha f_\beta}{f_\alpha^2} f_{\alpha\beta} s_{ij} s^i_0 \end{aligned} \tag{2.10}$$

Contracting (2.10) with $y^j$ and using $\alpha f_{\alpha\beta} = -\beta f_{\beta\beta}$ and $\alpha_{.i} s^i_{0|j} y^j = 0$, we get

$$y^j r_{.i} s^i_{0|j} = -\frac{f f_{\beta\beta}}{f_\alpha^2} s^i_0 s_{i0}. \tag{2.11}$$

Substituting (2.8) and (2.11) into (2.5), we get the result.

**Lemma 2.3.** $B^i_{.j} B^j_{.i} = -2rA s^i_0 s_{i0} - r^2 s^{ij} s_{ij}$, where $A = \frac{f_\alpha f_\beta + f f_{\alpha\beta}}{\alpha f_\alpha^2}$.

**Proof.**

$$\begin{aligned} B^i_{.j} B^j_{.i} &= (r_{.j} s^i_0 + r s^i_j)(r_{.i} s^j_0 + r s^j_i) \\ &= r_{.j} r s^i_0 s^j_i + r r_{.i} s^i_j s^j_0 + r_{.j} r_{.i} s^i_0 s^j_0 + r^2 s^i_j s^j_i. \end{aligned} \tag{2.12}$$

Using the relation $\alpha_{.j}s^i_j = \frac{1}{\alpha}s_{0i}$ and a direct computation, we get

$$r_{.j}s^i_0 s^j_i = r_{.i}s^j_0 s^i_j = -As^i_0 s_{i0}. \tag{2.13}$$

Using the relations $\alpha_i s^i_0 = 0$ and $b_i s^i_0 = 0$, we get

$$r_{.j}r_{.i}s^i_0 s^j_0 = 0. \tag{2.14}$$

Plugging (2.13) and (2.14) into (2.12), we get the result.

**Lemma 2.4.** $B^i_{\ |i} = rs^i_{\ 0|i} - \frac{ff_{\beta\beta}}{f^2_\alpha}s^i_{\ 0}s_{i0}$.

**Proof.** It follows from (2.6) that

$$B^i_{\ |i} = r_{|i}s^i_0 + rs^i_{0|i} = \frac{ff_{\beta\beta}}{f^2_\alpha}s_{0i}s^i_0 + rs^i_{0|i} = rs^i_{0|i} - \frac{ff_{\beta\beta}}{f^2_\alpha}s^i_{\ 0}s_{i0}$$

Summarizing up, we get the proof of Theorem 1.1. Q.E.D.

Let us remark that if we rewrite $F = f(\alpha,\beta)$ as $F = \alpha\phi(s)$, where $s = \frac{\alpha}{\beta}$, then we have

$$f_\alpha = \phi - s\phi', f_\beta = \phi', f_{\alpha\beta} = -s\alpha^{-1}\phi'', f_{\beta\beta} = \alpha^{-1}\phi''.$$

Consequently, (1.7) is rewritten as follows

$$R^i_i = \bar{R}^i_{\ i} + 2\alpha Q s^i_{\ 0|i} + \alpha^2 Q^2 s^{ij}s_{ij} + 2\,\Xi s^i_{\ 0}s_{i0}. \tag{2.15}$$

where $Q := \frac{\phi'}{\phi - s\phi'}$ and $\Xi := \frac{\phi\phi'^2 - s\phi'^3 - \phi^2\phi''}{(\phi - s\phi')^3}$.

If $\phi = 1 + s$ or equivalently $F = \alpha + \beta$ is a Randers metric with Killing 1-form with constant length, then we have

$$R^i_i = \bar{R}^i_i + 2\alpha s^i_{0|i} + \alpha^2 s^{ij}s_{ij} + 2s^i_{\ 0}s_{i0}. \tag{2.16}$$

**Corollary 2.5.** A Randers metric $F = \alpha + \beta$ with Killing 1-form with constant length is Ricci-quadratic if and only if

$$s^i_{0|i} = 0 \tag{2.17}$$

Let us recall that in Bao-Shen's sphere Randers metric $F = \alpha + \beta$, the 1-form $\beta$ is a Killing 1-form with constant length [6]. It is proved that $F$ has non-zero constant flag curvature. Hence, it is an Einstein Randers metric. We know that if an Einstein Randers metric is Ricci-quadratic, then it is either Riemannian metric or Ricci-flat. Hence, Bao-Shen's sphere Randers metric is not Ricci-quadratic, because it is neither Riemannian metric nor Ricci-flat.

**Example 1.** The projective spherical metric on $R^3$ is given by the following:

$$\alpha := \frac{\sqrt{(1 + ||X||^2)||Y||^2 - < X, Y >^2}}{1 + < X, X >}, \ X \in R^3, \ Y \in T_X R^3$$

where $<,>$ and $||.||$ denote the Euclidean inner product and norm on $R^3$, respectively. Put $X = (x, y, z)$ and $Y = (u, v, w)$. Suppose that $\beta = b_1 u + b_2 v + b_3 w$ is a Killing 1-form of $\alpha$. It is proved that

$$b_1 = \frac{1}{1 + < X, X >}(Q^1_2 y + Q^1_3 z + C^1),$$

$$b_2 = \frac{1}{1 + < X, X >}(Q^2_1 x + Q^2_3 z + C^2),$$

$$b_3 = \frac{1}{1 + < X, X >}(Q^3_1 x + Q^3_2 y + C^3),$$

where $Q = (Q^i_j)$ is an antisymmetric real matrix and $C = (C^i)$ is a constant vector in $R^3$. Let $C = (0, 1, 0)$ and $Q^1_2 = Q^2_3 = 0, Q^1_3 = 1$. Using a Maple program shows that in this case, $\beta$ is a Killing 1-form with unit length with respect to $\alpha$, which is not a closed 1-form.

## 3. Proof of Theorem 1.2

In this section, as an application of Theorem 1.1, we deal with the metric $F = \alpha + a\beta + b\,\frac{\beta^2}{\alpha}$, where $a$ and $b$ are two non-zero real constant. This class of Finsler metrics contains the class of Randers metrics as a special case.

Plugging (1.8) into (1.7), led to the following equation

$$Rat + 2a\alpha Irrat = 0, \tag{3.1}$$

where

$$\begin{aligned} Rat := {} & (a^2 s^{ij} s_{ij} - Ric)\alpha^{10} + (bRic\,\beta^2 + \overline{R}^i_i - a^2 Ric\,\beta^2 \\ & +4b^2 s^{ij} s_{ij}\beta^2 + 2a^2 s^i_0 s_{i0} - a^2 b s^{ij} s_{ij}\beta^2 + 4bs^i_{0|i}\beta - 4bs^i_0 s_{i0})\alpha^8 \\ & +(-3b\overline{R}^i_i\beta^2 - 4b^3 s^{ij} s_{ij}\beta^4 - 6a^2 b s^i_0 s_{i0}\beta^2 - 8b^2 s^i_{0|i}\beta^3 + 2b^2 Ric\,\beta^4 \\ & +3a^2 bRic\,\beta^4)\alpha^6 + (3b^2\overline{R}^i_i\beta^4 + 4b^3 s^i_{0|i}\beta^5 - 3a^2 b^2 Ric\,\beta^6 - 2b^3 Ric\,\beta^6 \\ & -12b^3 s^i_0 s_{i0}\beta^4)\alpha^4 + (-b^4 Ric\,\beta^8 + a^2 b^3 Ric\,\beta^8 - b^3\overline{R}^i_i\beta^6)\alpha^2 \\ & +b^5 Ric\,\beta^{10}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} Irrat := {} & (s^i_{0|i} + 2bs^{ij} s_{ij}\beta - Ric\,\beta)\alpha^8 + (2bRic\,\beta^3 - 2b^2 s^{ij} s_{ij}\beta^3 - 2bs^i_{0|i}\beta^2)\alpha^6 \\ & +(-8b^2 s^i_0 s_{i0}\beta^3 + b^2 s^i_{0|i}\beta^4)\alpha^4 - 2b^3 Ric\,\beta^7\alpha^2 + b^4 Ric\,\beta^9. \end{aligned} \tag{3.3}$$

It is easy to see that $F$ is Einstein metric if and only if $Rat = 0$ and $Irrat = 0$. We can rewrite $Rat = 0$, as follows:

$$\mu\alpha^2 + b^5 Ric\,\beta^{10} = 0, \tag{3.4}$$

where

$$\begin{aligned} \mu := {} & (a^2 s^{ij} s_{ij} - Ric)\alpha^8 + (bRic\,\beta^2 + \overline{R}^i_i - a^2 Ric\,\beta^2 \\ & +4b^2 s^{ij} s_{ij}\beta^2 + 2a^2 s^i_0 s_{i0} - a^2 b s^{ij} s_{ij}\beta^2 + 4bs^i_{0|i}\beta - 4bs^i_0 s_{i0})\alpha^6 \\ & +(-3b\overline{R}^i_i\beta^2 - 4b^3 s^{ij} s_{ij}\beta^4 - 6a^2 b s^i_0 s_{i0}\beta^2 - 8b^2 s^i_{0|i}\beta^3 + 2b^2 Ric\,\beta^4 \\ & +3a^2 bRic\,\beta^4)\alpha^4 + (3b^2\overline{R}^i_i\beta^4 + 4b^3 s^i_{0|i}\beta^5 - 3a^2 b^2 Ric\,\beta^6 - 2b^3 Ric\,\beta^6 \\ & -12b^3 s^i_0 s_{i0}\beta^4)\alpha^2 - b^4 Ric\,\beta^8 + a^2 b^3 Ric\,\beta^8 - b^3\overline{R}^i_i\beta^6 \end{aligned} \tag{3.5}$$

It means that $\alpha^2$ divides $Ric(x)$, which is impossible unless $Ric(x) = 0$ or $\beta = 0$. In each case, we must have $\mu = 0$. If $\beta = 0$, then the proof is done. Suppose that $F$ is Ricci-flat, i.e., $Ric = 0$. Then the equation $\mu = 0$ is reduced to the following

$$\lambda\alpha^2 - b^3\overline{R}^i_i\beta^6 = 0, \tag{3.6}$$

where

$$\begin{aligned} \lambda := {} & a^2 s^{ij} s_{ij}\alpha^6 + (\overline{R}^i_i + 4b^2 s^{ij} s_{ij}\beta^2 + 2a^2 s^i_0 s_{i0} - a^2 b s^{ij} s_{ij}\beta^2 + 4bs^i_{0|i}\beta \\ & -4bs^i_0 s_{i0})\alpha^4 + (-3b\overline{R}^i_i\beta^2 - 4b^3 s^{ij} s_{ij}\beta^4 - 6a^2 b s^i_0 s_{i0}\beta^2 - 8b^2 s^i_{0|i}\beta^3)\alpha^2 \\ & +3b^2\overline{R}^i_i\beta^4 + 4b^3 s^i_{0|i}\beta^5 - 12b^3 s^i_0 s_{i0}\beta^4 \end{aligned} \tag{3.7}$$

Equation (3.6) implies that $\alpha^2$ divides $\bar{R}^i_i$, i.e., $\bar{R}^i_i = c\alpha^2$ where $c$ is a constant by Riemannian Schur lemma. This means that $\alpha$ is an Einstein metric. This completes the proof. Q.E.D.

**Remark 3.1.** Let $\alpha$ be the projective spherical metric on $R^3$ and $\beta = \lambda(zdx + dy - xdz)$, where $\lambda = \frac{1}{1+||X||^2}$. The Riemannian metric $\alpha$ is of constant curvature $K = 1$. Thus $\alpha$ is Einstein metric and $\bar{R}^i_i = 2\alpha^2$. A direct computation, using a Maple program shows that $F = \alpha + a\beta + b\,\frac{\beta^2}{\alpha}$ is not Einstein metric for any $a$ and $b$.

Therefore, the converse of Theorem 1.2 is not true.

## 4. Appendix

**Maple program of converse of theorem 1.2**

```
> alpha := sqrt((x^2+y^2+z^2+1)*(u^2+v^2+w^2)-(u*x+v*y+w*z)^2)/(x^2+y^2+z^2+1):
> a_11:=simplify(diff(alpha^2/2,u,u)):
> a_12:=simplify(diff(alpha^2/2,u,v)):
> a_13:=simplify(diff(alpha^2/2,u,w)):
> a_21:=a_12:
> a_22:=simplify(diff(alpha^2/2,v,v)):
> a_23:=simplify(diff(alpha^2/2,v,w)):
> a_31:=a_13:
> a_32:=a_23:
> a_33:=simplify(diff(alpha^2/2,w,w)):
> C1 := 0; -1; C2 := 1; -1; C3 := 0; -1; Q1_3 := 1; -1; Q1_2 := 0; -1; Q2_3 := 0:
> b1:=Q1_2*y+Q1_3*z+C1+(C1*x+C2*y+C3*z)*x:
> b2:=-Q1_2*x+Q2_3*z+C2+(C1*x+C2*y+C3*z)*y:
> b3:=-Q1_3*x-Q2_3*y+C3+(C1*x+C2*y+C3*z)*z:
> b_1 := simplify(a_11*b1+a_12*b2+a_13*b3):
> b_2 := simplify(a_21*b1+a_22*b2+a_23*b3):
> b_3 := simplify(a_31*b1+a_32*b2+a_33*b3):
> Closeness12:=simplify(diff(b_1,y)-diff(b_2,x)):
> Closeness13:=simplify(diff(b_1,z)-diff(b_3,x)):
> Closeness23:=simplify(diff(b_2,z)-diff(b_3,y)):
> beta:=b_1*u+b_2*v+b_3*w:
```

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In this part we want to find r_ij and s_ij ;

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**a_ij:=Matrix([[a_11,a_12,a_13],[a_21,a_22,a_23],[a_31,a_32,a_33]]):**

```
> R1 := LinearAlgebra:-MatrixInverse(a_ij):
> a11 := R1['1, 1']:
> a12 := R1['1, 2']:
> a13 := R1['1, 3']:
> a21 := R1['2, 1']:
> a22 := R1['2, 2']:
> a23 := R1['2, 3']:
> a31 := R1['3, 1']:
> a32 := R1['3,2']:
> a33:= R1['3, 3']:
> b1:=simplify(a11*b_1+a12*b_2+a13*b_3):
> b2:=simplify(a21*b_1+a22*b_2+a23*b_3):
> b3:=simplify(a31*b_1+a32*b_2+a33*b_3):
> B2:=simplify(b1*b_1+b2*b_2+b3*b_3):
```

> **Here, we compute the Christofell symbols of the Riemannian metic "alpha:**

```
>  G1_11:=(1/2)*simplify(a11*(diff(a_11,x)+diff(a_11,x)-diff(a_11,x))+a12*(diff(a_21,x)+diff(a_12,x)
>  -diff(a_11,y))+a13*(diff(a_31,x)+diff(a_13,x)-diff(a_11,z))):
>  G1_12:=(1/2)*simplify(a11*(diff(a_11,y)+diff(a_21,x)-diff(a_12,x))+a12*(diff(a_21,y)+diff(a_22,x)
>  -diff(a_12,y))+a13*(diff(a_31,y)+diff(a_23,x)-diff(a_12,z))):
>  G1_13:=(1/2)*simplify(a11*(diff(a_11,z)+diff(a_31,x)-diff(a_13,x))+a12*(diff(a_21,z)+diff(a_32,x)
>  -diff(a_13,y))+a13*(diff(a_31,z)+diff(a_33,x)-diff(a_13,z))):
>  G1_21:=G1_12:
>  G1_22:=(1/2)*simplify(a11*(diff(a_12,y)+diff(a_21,y)-diff(a_22,x))+a12*(diff(a_22,y)
>  +diff(a_22,y)-diff(a_22,y))+a13*(diff(a_32,y)+diff(a_23,y)-diff(a_22,z))):
>  G1_23:=(1/2)*simplify(a11*(diff(a_12,z)+diff(a_31,y)-diff(a_23,x))+a12*(diff(a_22,z)+diff(a_32,y)
>  -diff(a_23,y))+a13*(diff(a_32,z)+diff(a_33,y)-diff(a_23,z))):
>  G1_31:=G1_13:
>  G1_32:=G1_23:
>  G1_33:=(1/2)*simplify(a11*(diff(a_13,z)+diff(a_31,z)-diff(a_33,x))+a12*(diff(a_23,z)+diff(a_32,z)
>  -diff(a_33,y))+a13*(diff(a_33,z)+diff(a_33,z)-diff(a_33,z))):
>  G2_11:=(1/2)*simplify(a12*(diff(a_11,x)+diff(a_11,x)-diff(a_11,x))+a22*(diff(a_12,x)+diff(a_21,x)
>  -diff(a_11,y))+a32*(diff(a_13,x)+diff(a_31,x)-diff(a_11,z))):
>  G2_12:=(1/2)*simplify(a12*(diff(a_12,x)+diff(a_11,y)-diff(a_12,x))+a22*(diff(a_22,x)+diff(a_21,y)
>  -diff(a_12,y))+a32*(diff(a_23,x)+diff(a_31,y)-diff(a_12,z))):
>  G2_13:=(1/2)*simplify(a12*(diff(a_31,x)+diff(a_11,z)-diff(a_13,x))+a22*(diff(a_32,x)+diff(a_21,z)
>  -diff(a_13,y))+a32*(diff(a_33,x)+diff(a_31,z)-diff(a_13,z))):
>  G2_21:=G2_12:
>  G2_22:=(1/2)*simplify(a12*(diff(a_21,y)+diff(a_12,y)-diff(a_22,x))+a22*(diff(a_22,y)+diff(a_22,y)
>  -diff(a_22,y))+a32*(diff(a_23,y)+diff(a_32,y)-diff(a_22,z))):
>  G2_23:=(1/2)*simplify(a12*(diff(a_31,y)+diff(a_12,z)-diff(a_23,x))+a22*(diff(a_32,y)+diff(a_22,z)
>  -diff(a_23,y))+a32*(diff(a_33,y)+diff(a_32,z)-diff(a_23,z))):
>  G2_31:=G2_13:
>  G2_32:=G2_23:
>  G2_33:=(1/2)*simplify(a12*(diff(a_31,z)+diff(a_13,z)-diff(a_33,x))+a22*(diff(a_32,z)+diff(a_23,z)
>  -diff(a_33,y))+a32*(diff(a_33,z)+diff(a_33,z)-diff(a_33,z))):
>  G3_11:=(1/2)*simplify(a31*(diff(a_11,x)+diff(a_11,x)-diff(a_11,x))+a32*(diff(a_21,x)+diff(a_12,x)
>  -diff(a_11,y))+a33*(diff(a_31,x)+diff(a_13,x)-diff(a_11,z))):
>  G3_12:=(1/2)*simplify(a31*(diff(a_11,y)+diff(a_21,x)-diff(a_12,x))+a32*(diff(a_21,y)+diff(a_22,x)
>  -diff(a_12,y))+a33*(diff(a_31,y)+diff(a_23,x)-diff(a_12,z))):
>  G3_13:=(1/2)*simplify(a31*(diff(a_11,z)+diff(a_31,x)-diff(a_13,x))+a32*(diff(a_21,z)+diff(a_32,x)
>  -diff(a_13,y))+a33*(diff(a_31,z)+diff(a_33,x)-diff(a_13,z))):
>  G3_21:=G3_12:
>  G3_22:=(1/2)*simplify(a31*(diff(a_12,y)+diff(a_21,y)-diff(a_22,x))+a32*(diff(a_22,y)+diff(a_22,y)
>  -diff(a_22,y))+a33*(diff(a_32,y)+diff(a_23,y)-diff(a_22,z))):
>  G3_23:=(1/2)*simplify(a31*(diff(a_12,z)+diff(a_31,y)-diff(a_23,x))+a32*(diff(a_22,z)+diff(a_32,y)
>  -diff(a_23,y))+a33*(diff(a_32,z)+diff(a_33,y)-diff(a_23,z))):
>  G3_31:=G3_13:
>  G3_32:=G3_23:
>  G3_33:=(1/2)*simplify(a31*(diff(a_13,z)+diff(a_31,z)-diff(a_33,x))+a32*(diff(a_23,z)+diff(a_32,z)
>  -diff(a_33,y))+a33*(diff(a_33,z)+diff(a_33,z)-diff(a_33,z))):
```

In this part we will compute the geodesic sprays and projective quantities of the alpha :

```
>  G1_a:=1/2*simplify(G1_11u^2+G1_12uv+G1_13uw+G1_21vu+G1_22v^2+G1_23vw+G1_31wu+G1_32wv+G1_33w^2):
>  G2_a:=1/2*simplify(G2_11u^2+G2_12uv+G2_13uw+G2_21vu+G2_22v^2+G2_23vw+G2_31wu+G2_32wv+G2_33w^2):
>  G3_a:=1/2*simplify(G3_11u^2+G3_12uv+G3_13uw+G3_21vu+G3_22v^2+G3_23vw+G3_31wu+G3_32wv+G3_33w^2):
```

Here, we have Gi_a=Py^i, where P=-(xu+yv+zw)/(1+x^2+y^2+z^2).Hence, “alpha is projectively flat. Therefore “alpha is of constant curavture by Beltrami theorem.

Every Riemannian metric with constant curavture is Einstein metric. Thus “alpha is Einstein metric.

```
>  P := G1_a/u:
>  Xi := simplify(P^2-(diff(P, x))*u-(diff(P, y))*v-(diff(P, z))*w):
>  K := simplify(Xi/alpha^2):
```

This meants that alpha is of constant flag curvature K=1. Hence, we have $\bar{R}_i^i = 2\alpha^2$.
Now we are going to compute b_i—j =diff(b_i,x_j)-Gk_ij*b_k :

```
>  b_11:=simplify(diff(b_1,x)-G1_11*b_1-G2_11*b_2-G3_11*b_3):
```

```
> b_12:=simplify(diff(b_1,y)-G1_12*b_1-G2_12*b_2-G3_12*b_3):
> b_13:=simplify(diff(b_1,z)-G1_13*b_1-G2_13*b_2-G3_13*b_3):
> b_21:=simplify(diff(b_2,x)-G1_21*b_1-G2_21*b_2-G3_21*b_3):
> b_22:=simplify(diff(b_2,y)-G1_22*b_1-G2_22*b_2-G3_22*b_3):
> b_23:=simplify(diff(b_2,z)-G1_23*b_1-G2_23*b_2-G3_23*b_3):
> b_31:=simplify(diff(b_3,x)-G1_31*b_1-G2_31*b_2-G3_31*b_3):
> b_32:=simplify(diff(b_3,y)-G1_32*b_1-G2_32*b_2-G3_32*b_3):
> b_33:=simplify(diff(b_3,z)-G1_33*b_1-G2_33*b_2-G3_33*b_3):
```

We know that s_ij:=1/2(b_ij-b_ji):

```
> s_11 := 0; -1; s_22 := 0; -1; s_33 := 0:
> s_12:=simplify((b_12-b_21)/2):s_21:=-s_12:simplify(2*s_12-Closeness12):
> s_13:=simplify((b_13-b_31)/2):  s_31:=-s_13:simplify(2*s_13-Closeness13):
> s_23:=simplify((b_23-b_32)/2):s_32:=-s_23:  simplify(2*s_23-Closeness23):
> s_10:=simplify((s_11u+s_12v+s_13w)):
> s_20:=simplify((s_21u+s_22v+s_23w)):
> s_30:=simplify((s_31u+s_32v+s_33w)):
```

r_ij:=1/2(b_ij+b_ji).

```
> r_11:=b_11:
> r_12:=(b_12+b_21)/2:
> r_13:=(b_13+b_31)/2:
> r_21:=(b_21+b_12)/2:
> r_22:=b_22:
> r_23:=(b_23+b_32)/2:
> r_31:=(b_31+b_13)/2:
> r_32:=(b_32+b_23)/2:
> r_33:=b_33:
```

**Here, we compute**

```
>  Si_0S_i0
> s1_1:=a11*s_11+a12*s_21+a13*s_31:
> s1_2:=a11*s_12+a12*s_22+a13*s_32:
> s1_3:=a11*s_13+a12*s_23+a13*s_33:
> s2_1:=a21*s_11+a22*s_21+a23*s_31:
> s2_2:=a21*s_12+a22*s_22+a23*s_32:
> s2_3:=a21*s_13+a22*s_23+a23*s_33:
> s3_1:=a31*s_11+a32*s_21+a33*s_31:
> s3_2:=a31*s_12+a32*s_22+a33*s_32:
> s3_3:=a31*s_13+a32*s_23+a33*s_33:
> s1_0:=a1_1*u+a1_2*v+a1_3w:
> s2_0:=a2_1*u+a2_2*v+a2_3w:
> s3_0:=a3_1*u+a3_2*v+a3_3w:
```

**Here, we compute**

```
>  SijS_ij
> s11:=a11*s1_1+a12*s1_2+a13*s1_3:
> s12 := a21*s1_1+a22*s1_2+a23*s1_3:
> s13 := a31*s1_1+a32*s1_2+a33*s1_3:
```

```
> s21 := a11*s2_1+a12*s2_2+a13*s2_3:
> s22 := a21*s2_1+a22*s2_2+a23*s2_3:
> s23 := a31*s2_1+a32*s2_2+a33*s2_3:
> s31 := a11*s3_1+a12*s3_2+a13*s3_3
> s32 := a21*s3_1+a22*s3_2+a23*s3_3:
> s33 := a31*s3_1+a32*s3_2+a33*s3_3:
> SijS_ij := simplify(s11*s_11+s12*s_12+s13*s_13+s21*s_21+s22*s_22+s23*s_23+s31*s_31+s32*s_32+s33*s_33):
```

Here, we compute Si_0i

```
> s1_11:=simplify(diff(s1_1,x)+s1_1*G1_11+s2_1*G1_21+s3_1*G1_31-s1_1*G1_11-s1_2*G2_11-s1_3*G3_11):
> s1_12 := simplify(diff(s1_1, y)+s2_1*G1_22+s3_1*G1_32-s1_2*G2_12-s1_3*G3_12):
> s1_13 := simplify(diff(s1_1, z)+s2_1*G1_23+s3_1*G1_33-s1_2*G2_13-s1_3*G3_13):
> s1_21:=simplify(diff(s1_2,x)+s1_2*G1_11+s2_2*G1_21+s3_2*G1_31-s1_1*G1_21-s1_2*G2_21-s1_3*G3_21):
> s1_22 := simplify(diff(s1_2, y)+s1_2*G1_12+s2_2*G1_22+s3_2*G1_32-s1_1*G1_22-s1_2*G2_22-s1_3*G3_22):
> s1_23 := simplify(diff(s1_2, z)+s1_2*G1_13+s2_2*G1_23+s3_2*G1_33-s1_1*G1_23-s1_2*G2_23-s1_3*G3_23);:
> s1_31:=simplify(diff(s1_3,x)+s1_3*G1_11+s2_3*G1_21+s3_3*G1_31-s1_1*G1_31-s1_2*G2_31-s1_3*G3_31):
> s1_32 := simplify(diff(s1_3, y)+s1_3*G1_12+s2_3*G1_22+s3_3*G1_32-s1_1*G1_32-s1_2*G2_32-s1_3*G3_32):
> s1_33 := simplify(diff(s1_3, z)+s1_3*G1_13+s2_3*G1_23+s3_3*G1_33-s1_1*G1_33-s1_2*G2_33-s1_3*G3_33):
> s2_11:=simplify(diff(s2_1,x)+s1_1*G2_11+s2_1*G2_21+s3_1*G2_31-s2_1*G1_11-s2_2*G2_11-s2_3*G3_11):
> s2_12 := simplify(diff(s2_1, y)+s1_1*G2_12+s2_1*G2_22+s3_1*G2_32-s2_1*G1_12-s2_2*G2_12-s2_3*G3_12):
> s2_13 := simplify(diff(s2_1, z)+s1_1*G2_13+s2_1*G2_23+s3_1*G2_33-s2_1*G1_13-s2_2*G2_13-s2_3*G3_13):
> s2_21:=simplify(diff(s2_2,x)+s1_2*G2_11+s2_2*G2_21+s3_2*G2_31-s2_1*G1_21-s2_2*G2_21-s2_3*G3_21):
> s2_22 := simplify(diff(s2_2, y)+s1_2*G2_12+s3_2*G2_32-s2_1*G1_22-s2_3*G3_22):
> s2_23 := simplify(diff(s2_2, z)+s1_2*G2_13+s3_2*G2_33-s2_1*G1_23-s2_3*G3_23):
> s2_31:=simplify(diff(s2_3,x)+s1_3*G2_11+s2_3*G2_21+s3_3*G2_31-s2_1*G1_31-s2_2*G2_31-s2_3*G3_31):
> s2_32 := simplify(diff(s2_3, y)+s1_3*G2_12+s2_3*G2_22+s3_3*G2_32-s2_1*G1_32-s2_2*G2_32-s2_3*G3_32):
> s2_33 := simplify(diff(s2_3, z)+s1_3*G2_13+s2_3*G2_23+s3_3*G2_33-s2_1*G1_33-s2_2*G2_33-s2_3*G3_33):
> s3_11:=simplify(diff(s3_1,x)+s1_1*G3_11+s2_1*G3_21+s3_1*G3_31-s3_1*G1_11-s3_2*G2_11-s3_3*G3_11):
> s3_12 := simplify(diff(s3_1, y)+s1_1*G3_12+s2_1*G3_22+s3_1*G3_32-s3_1*G1_12-s3_2*G2_12-s3_3*G3_12):
> s3_13 := simplify(diff(s3_1, z)+s1_1*G3_13+s2_1*G3_23+s3_1*G3_33-s3_1*G1_13-s3_2*G2_13-s3_3*G3_13):
> s3_21:=simplify(diff(s3_2,x)+s1_2*G3_11+s2_2*G3_21+s3_2*G3_31-s3_1*G1_21-s3_2*G2_21-s3_3*G3_21):
> s3_22 := simplify(diff(s3_2, y)+s1_2*G3_12+s2_2*G3_22+s3_2*G3_32-s3_1*G1_22-s3_2*G2_22-s3_3*G3_22):
> s3_23 := simplify(diff(s3_2, z)+s1_2*G3_13+s2_2*G3_23+s3_2*G3_33-s3_1*G1_23-s3_2*G2_23-s3_3*G3_23):
> s3_31:=simplify(diff(s3_3,x)+s1_3*G3_11+s2_3*G3_21+s3_3*G3_31-s3_1*G1_31-s3_2*G2_31-s3_3*G3_31):
> s3_32 := simplify(diff(s3_3, y)+s1_3*G3_12+s2_3*G3_22-s3_1*G1_32-s3_2*G2_32):
> s3_33 := simplify(diff(s3_3, z)+s1_3*G3_13+s2_3*G3_23-s3_1*G1_33-s3_2*G2_33):
> s1_01 := simplify(s1_11*u+s1_21*v+s1_31*w):
> s2_02 := simplify(s2_12*u+s2_22*v+s2_32*w):
> s3_03 := simplify(s3_13*u+s3_23*v+s3_33*w):
> Si_0i := simplify(s1_01+s2_02+s3_03):
> F:=alpha+a*beta+b*(beta)^2/alpha:


> r:=simplify(-alpha^2(a*alpha+2b*beta)/(-alpha^2+b*beta^2)):
> t:= simplify(2alpha^2(3a^2alpha^2b*beta^2+6b^3beta^4-a^2alpha^4+8a*alpha b^2 beta^2+2alpha^4b)
> /(b*beta^2-alpha^2)^3)
> Q:=simplify(2rSi_0i+r^2SijS_ij+tSi_0S_i0):
```

```
> R^i_i := simplify((2*alpha^2+Q)/F^2):
```

If R^i_i is a function of only (x,y,z), then F is an Einstein metric. But this is not the case. Hence, F is not Einstein metric!!

## References


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